Knott, Sang and Spence on logarithmic and other tables

Alex D.D. Craik,
Emeritus Professor,
School of Mathematics & Statistics,
University of St Andrews,
St Andrews, KY16 9SS,
Scotland, UK
addc@st-andrews.ac.uk

ABSTRACT: This paper summarises the lives and work of three Scottish mathematicians who were involved in logarithmic and other tables. They are, in reverse chronological order, Cargill Gilston Knott (1856-1922), Edward Sang (1805-1890), and William Spence (1777-1815).

1. Cargill Gilston Knott

The Napier Tercentenary meeting, held in Edinburgh in 1914 just before the outbreak of war, was a very grand and extensive affair. The present celebration of the 400th anniversary of the publication of John Napier's *Mirifici Logarithmorum Canonis Descriptio* is comparatively low-key. This doubtless reflects the profound changes in mathematical and scientific research over the last one hundred years. In place of extensive construction and use of tables of mathematical functions and the operation of mechanical calculating instruments, electronic computers and powerful pocket calculators now reign supreme. The use of logarithmic tables in simplifying arithmetical calculations is now taught, if at all, in courses on the history of mathematics.

A main organiser of the extensive Napier Tercentenary meeting in 1914 was Cargill Gilston Knott, and it seems appropriate to remember him here today. When Knott graduated from Edinburgh University in 1876, he became a research assistant to Peter Guthrie Tait, Edinburgh's professor of natural philosophy. There, he worked "in an ill-equipped attic," receiving in 1879 a D.Sc. for his researches on contact electricity. In 1883, Knott played a leading part in the establishment of the Edinburgh Mathematical Society, becoming its first secretary and treasurer; but he left that same year to become professor of physics at the Imperial University of Tokyo, a post he held until 1891. Not long after his arrival, the Tokyo Mathematical Society, founded in 1877, was renamed the *Tokyo Mathematico-Physical Society*, and in this Knott played a significant part.

On returning to Edinburgh, Knott was appointed by Tait to a newly-created lectureship, and he was later promoted to reader but never to professor. He had probably hoped to become Tait's successor in 1901; but the job went instead to another of Tait's former students, Canadian-born James MacGregor, who had been professor of physics at Dalhousie University, Nova Scotia. Yet Knott's scientific output and services to education surpassed those of many university professors. His relative lack of advancement must have been associated with a desire to remain in Edinburgh.

Knott's main teaching duties at Edinburgh were in applied mathematics, on which he gave a regular fifty-lecture course. From 1906, he was also in charge of new teaching laboratories. An active fellow of the Royal Society of Edinburgh, he served for many years as Council member and

General Secretary; he twice served years as President of the Edinburgh Mathematical Society; and he was President of the Scottish Meteorological Society. In 1916, he was awarded an honorary LL.D. by the University of St Andrews, and he was elected FRS in 1920, just two years before his death.

When in Japan, Knott was attracted to ongoing researches on seismology, and he gave these a sounder physical and theoretical basis. As well as specialist papers, he published a general work, *The Physics of Earthquake Phenomena* (1908), based on a lecture course delivered, rather surprisingly, at the United Free Church College in Aberdeen. (Knott had close Free Church affiliations, and some of his family were missionaries.) But Knott's major research was undoubtedly that on magnetism. In 1887, he organised a major magnetic survey of Japan, helped by a group of students, including Nagaoka Hantaro and Tanakadate Aikitsu who became leaders of Japanese science.

Knott was also an advocate of *quaternions*, the mathematical system invented by William Rowan Hamilton and popularised by P.G. Tait as a natural notation for much of mathematical physics. Knott inherited Tait's mantle as the main proponent of this system, arguing against the supporters of vector calculus who eventually prevailed.

Knott was a prolific author, with scientific papers mainly on magnetism, earthquakes and quaternions; articles for the *Encyclopaedia Britannica* and *Chambers' Encyclopaedia*; magazine articles on Japanese themes; and further books on *Electricity and Magnetism* (1893) and *Physics* (*Elementary*) (1897). He prepared a revised edition of Kelland and Tait's *Quaternions* (3rd edn. 1904); and he edited a booklet of *Four-figure Mathematical Tables* much used in Scottish schools until the 1970s. His *Memoir of Professor P.G. Tait* (1911) is the standard biographical source, prepared as a supplement to Tait's collected scientific papers; and he edited the *Collected Scientific Papers of the Late Dr John Aitken, F.R.S.* (1922) and a book on *Edinburgh's Place in Scientific Progress* (1921).

For the 1914 meeting to mark the three-hundredth anniversary of the publication of Napier's *Mirifici Logarithmorum Canonis Descriptio,* Knott tried hard, but unsuccessfully, to revive plans to publish the mammoth logarithmic and other tables constructed by another former Edinburgh citizen, Edward Sang (1805-1890). He also edited the resulting *Napier Tercentenary Memorial Volume* (Knott, 1915), in which he published an article entitled "Edward Sang and his Logarithmic Calculations". Also, in the accompanying *Handbook of the Tercentenary Exhibition*… (Horsburgh, 1914), are reprinted two of his previous articles: "Dr Edward Sang's Logarithmic, Trigonometric, and Astronomical Tables" and "The Calculating Machine of the East: the Abacus". Further details of Knott's activities are in (Whittaker, 1922-23) and (Craik, 2007).

2. Edward Sang

Kirkaldy-born Edward Sang was a mathematical and computational prodigy. At the age of 12, he completed his schooling and was awarded a prize of Legendre's *Éléments de Géométrie* for his success in mathematics. There seems little doubt that he could and did read it. Entering Edinburgh University in 1818, he had to enrol in John Leslie's second class of mathematics, since the third, advanced, class was not taught in that year. Small for his age, and still just 13, he was first mocked by his fellow-students, but they were soon awed by his precocious talent. In 1819, Leslie took over the chair of Natural Philosophy and William Wallace was appointed as his replacement in mathematics: as a result, there was again no advanced class in mathematics. Despite periods of illness, the young Edward impressed both Leslie and Wallace.

On leaving university in 1824, Sang first worked in Edinburgh as a surveyor, civil engineer and mathematics teacher, and he lectured on natural philosophy (perhaps as John Leslie's assistant). He published several papers on disparate topics, and actively participated in Edinburgh's Royal

Scottish Society of Arts, to which he was elected a Fellow in 1828. By 1836 he was its vice-president, and delivered an "Annual Report on the State of the Useful Arts". In 1840, his *Essays on Life Assurance* were privately published. He had married in 1832 and had five children: two of his daughters, Jane and Flora, were later to help him in his compilation of mathematical tables.

Sang failed to succeed Wallace as Edinburgh's professor of mathematics, the post going to Philip Kelland, a Cambridge graduate and one of the first Englishmen to hold an Edinburgh chair. Sang then left Edinburgh for Manchester where, during 1841-43, he was Professor of Mechanical Sciences at the nonconformist Manchester New College. Sang had been elected FRSE in 1836, but his fellowship was cancelled in 1840, perhaps because he no longer wished to afford the subscription after leaving Edinburgh.

In 1843 Sang's career took a more dramatic turn, when he went to Constantinople to establish engineering schools, plan railways and an ironworks. He learned Turkish and lectured in that language at the Imperial School, Muhendis-hana Berii. While there, he gained fame by predicting the solar eclipse of 1847 and thereby dispelling local superstition. According to Sang's obituarist, D. Bruce Peebles (Peebles, 1897, xxii), his pupils prepared several textbooks based upon his lectures. In 1849, while still in Turkey, he was re-elected a Fellow of the Royal Society of Edinburgh. He was invited by the British Association for the Advancement of Science to go to Russia to observe the total solar eclipse of 1851; but, receiving too short notice and calm weather delaying his voyage, he arrived too late to make many useful observations. In the following year, his *A New General Theory of the Teeth of Wheels*, written in Constantinople, was published in Edinburgh.

Much against the wishes of the Sultan Abdul-Mejid I, Sang resigned his post in 1854. He may have been influenced by the fact that his daughters were approaching adulthood. But the main reason for his leaving Turkey was surely the deteriorating political situation: Russia had declared war on Turkey in October 1853, and the French and British fleets intervened on Turkey's behalf. The Crimean War began in September 1854, when Great Britain and France, later followed by Austria, declared war on Russia.

On his return to Edinburgh, Sang busied himself "as a teacher of mathematics, actuary and general consultant in a wide variety of subjects involving applied mathematics" (Davidson 1956, 26), with premises at 2 George Street. In 1859, he was an unsuccessful candidate for the Edinburgh Natural Philosophy Chair, when Tait won it over Clerk Maxwell.

In 1856, the Faculty of Actuaries in Scotland was established as a professional association, with the aim of promoting "a satisfactory school of actuarial study" (Davidson 1956, 33). Sang was involved from the start in the planning of qualifying examinations and a library; and in the first year he gave a course of four actuarial lectures. By 1859, the Faculty was fully operational, with a junior affiliated body, the Actuarial Society, providing for the needs of students. Sang, the first official lecturer to the Faculty, provided courses to the Society: in 1864, he published *A Treatise on the Valuation of Life Contingencies*, he reissued extended *Life Assurance and Annuity Tables*, and in 1868 he delivered an *Address to the Actuarial Society*. According to (Davidson, 1956, 29): "Sang's influence on the actuarial aspirants of his day was immediate and all-important... and from him and one or two others came the original impetus for actuarial study and original investigation in Scotland."

Sang published individualistic books on *Elementary* and *Higher Arithmetic* in 1856-1857. In the latter, he shows how to calculate and to use logarithms, working always to seven or more decimal places. Though most of the arithmetical procedures that he advocated never became common practice, there is no doubt that they worked well for himself, an arithmetician *par excellence*. It is also very likely that he taught these methods to his daughters Flora and Jane, who assisted him in his logarithmic calculations. Sang disapproved of rote learning, believing that subjects should not be taught abstractly, but demonstrated in relation to the practical arts and crafts and to aspects of everyday life. In particular, he was opposed to the rigid Euclidean approach to geometry. These views were very much in line with the ideals of the Royal Scottish Society of Arts, and of the various Mechanics Institutes established around this time to educate tradesmen and apprentices.

Mainly for Edinburgh-based journals, Sang wrote many papers on mathematical, mechanical, optical and actuarial topics. He built elegant physical apparatus, and he was a meticulous engineering draughtsman. One of his experiments is said to have anticipated Foucault's famous pendulum demonstration of the Earth's rotation. The harbour of Sang's hometown of Kirkcaldy was the beneficiary of his innovative dioptric light. Sang's publications, and many surviving unpublished manuscripts, cover an immense range of interests and a 60-year timespan. A few are on mathematics unconnected with any application; but the bulk concern topics in physics, astronomy and engineering, ranging through fly-wheels and turning lathes, vibrating wires, toothed wheels, clock mechanisms, meteorological observations, carpet manufacture, railways, achromatic optical lenses, construction of arches, surveying methods, the form of ships' hulls, actuarial calculations, and solar eclipses. He also wrote several articles for *Encyclopaedia Britannica*. His obituarist Bruce Peebles (Peebles, 1897) lists 112 works, and the *Royal Society Catalogue of Scientific Papers 1800-1900* gives the same number (though they disagree on titles). Rather late in life, Sang received various honours, though he was never a rich man. He received prizes from the Royal Scottish Society of Arts (1861), the Institution of Civil Engineers, London (1879) and the Royal Society of Edinburgh (1886). He was elected a corresponding member of the Royal Tunis Academy (1881), an honorary LL.D. of Edinburgh University (1883) and an honorary member of the Franklin Institute, Philadelphia (1884).

Astronomical conversion tables appeared three years before his published tables of 7-place logarithms (Sang, 1871). In 1872, he proposed the publication of a *Million Table of nine-place logarithms*, but this failed to get enough support. In 1882, William Swan, the President of the Royal Scottish Society of Arts, alluded to "the everlasting discredit of our country" in failing to provide the means to publish Sang's logarithmic tables. Sang died on 23 December 1890 in his 86th year. In a 1934 retrospect to commemorate the 150th year of the Royal Society of Edinburgh, its President, the biologist and polymath D'Arcy Wentworth Thompson, paid tribute to Sang, singling out Sang's manuscript logarithmic and other tables as among the Society's most prized possessions. It is these that we now discuss.

On 5 November, 1907, *The Scotsman* published an extract from the Minutes of the Council of the Royal Society of Edinburgh, which later appeared in its *Proceedings* (Knott, 1908) and was partially reprinted in Horsburgh's *Napier Exhibition Catalogue* (Horsburgh, 1914, 38-47)). This records the gift to the nation by the Misses Sang of Edward Sang's mathematical manuscripts (about which there had been protracted and at times acrimonious negotiations). The article gives a brief description of the tables:

> The manuscript volumes number forty-seven in all, the contents of thirty-three of which are in transfer duplicate [i.e. a total of 80 volumes]. Volumes 1 to 3 contain the details of the steps of the calculations on which the results contained in the next thirty-six volumes are based.
>
> Volume 4 contains the logarithms, calculated to 28 figures, of the prime numbers up to 10,000, and a few beyond.
>
> Volumes 5 and 6 contain the logarithms to 28 figures of all numbers up to 20,000. From these the succeeding thirty-two volumes are constructed, giving the logarithms to 15 places of all the numbers from 100,000 to 370,000.
>
> This colossal work must ever remain of the greatest value to computers of logarithmic tables. It is a great national possession.
>
> The other Tables in the collection are trigonometrical and astronomical. Of special interest are the Tables of Sines and Tangents calculated according to the centesimal division of the quadrant...

> ... In the name of the British Nation, the Royal Society of Edinburgh now publicly thank the Misses Sang for their valuable gift, and, as custodiers of these manuscript volumes, undertake to do all in their power to make them of real use to the scientific world. (Knott 1908, 183-4).

Sang's logarithmic tables are, of course, common logarithms to base 10. These manuscripts were exhibited at the Napier Tercentenary meeting. They exceed in accuracy and extent the tables of the French Bureau du Cadastre, produced by Gaspard de Prony and a multitude of assistants during 1794-1801. But, like Prony's, only a small part of Sang's tables was published: his 7-place logarithmic tables of 1871. The original volumes are now mostly in the National Library of Scotland and the transfer duplicates in Edinburgh University Library; but at some stage vols. 7-11 (*Logarithms 10-14*) were sent to the University, perhaps by mistake, where they remain.

Further to my own account of this material (Craik, 2003), a more recent study of Sang's tables, with a full re-computation and list of his publications, has been made by Denis Roegel of Nancy: see http://locomat.loria.fr/. Suffice to say that Sang's achievement in producing these tables (with help from two of his daughters, Flora and Jane) was a prodigious one, but now rendered irrelevant by electronic computational devices: if wanted, such tables can now be compiled at a speed limited only by that of the printer.

3. William Spence

Our third subject, William Spence, lived at an earlier time, when British mathematics was at a low ebb. He was brought up in Greenock, near Glasgow, and attended no university. For some years, he lodged in Glasgow with William Struthers, a banker and family friend, who had (it was said) a "profound knowledge of mathematics" and a large personal library. Spence returned to Greenock in 1797 and pursued his private researches, without any occupation. But for his friendship with the writer John Galt, he might well have disappeared from view.

His publications (with small print-runs) are:

1809: *Essay on the Theory of the Various Orders of Logarithmic Transcendents; with an inquiry into their applications to the integral calculus and the summation of series.*

1814/1817: *Outlines of a Theory of Algebraic Equations deduced from the Principles of Harriott, and extended to the Fluxional and Differential Calculus.*

Following Spence's early death, Galt commissioned John Herschel to edit his unpublished manuscripts:

1819: *Mathematical Essays of the late William Spence, Esq….,* ed. J.F.W. Herschel, with biographical memoir by J. Galt

which reprinted Spence's two previously-published works along with several manuscript pieces.

Spence was familiar with much continental literature, then largely ignored in Britain. Among authors cited in his *Logarithmic Transcendents* are Euler, Landen, Lorgna, l'Huillier, Ferroni, Laplace, Legendre and Lacroix. He wrote in his Preface (1809) that:

> … Our pupils are taught the science [of Analytical Mathematics] by means of the applications. On the Continent, Analysis is studied as an independent science. Let any

person who has studied Mathematics only in British authors look at the works of the higher analysis of the Continent, and he will soon perceive that he has still much to learn."

Spence's "logarithmic transcendents", now called *polylogarithms,* are a family of transcendental functions of which the first is the ordinary logarithm. These comprise one of the first families of transcendental functions to be investigated, though long overshadowed by what later became the "special functions" of classical mathematical physics and applied mathematics.

Spence's "logarithmic transcendents" are the set of functions $\overset{n}{\mathrm{L}}(1 \pm x)$ expressed as successive integrals (the first being just the hyperbolic logarithm of $1 \pm x$):

$$\overset{1}{\mathrm{L}}(1\pm x) = \int \frac{\pm dx}{1\pm x}, \quad \overset{2}{\mathrm{L}}(1\pm x) = \int \frac{dx}{x}\overset{1}{\mathrm{L}}(1\pm x), \quad \overset{3}{\mathrm{L}}(1\pm x) = \int \frac{dx}{x}\overset{2}{\mathrm{L}}(1\pm x),$$

$$\ldots\ldots \quad \overset{n}{\mathrm{L}}(1\pm x) = \int \frac{dx}{x}\overset{n-1}{\mathrm{L}}(1\pm x).$$

These have series expansions

$$\pm\ \frac{x}{1^n} - \frac{x^2}{2^n} \pm \frac{x^3}{3^n} - \ \&\mathrm{c.}$$

provided $|x| < 1$. (The integrals are taken from 0 to x: it was not yet customary to state limits on the integral signs.)

Spence ingeniously deduced many properties, and tabulated the functions $\overset{2}{\mathrm{L}}(1 + x)$ and $\overset{3}{\mathrm{L}}(1 + x)$, now called the *dilogarithm* and *trilogarithm* functions. He then showed how many integrals and series, that cannot be expressed in terms of algebraic and logarithmic functions, could be evaluated by means of these new functions.

John Herschel later wrote:

> A premature death carried off, in Spence, one who might have become the ornament of his country in this department of knowledge. His posthumous essays, which were not, however, collected and published till 1819, prove him to have been both a learned and inventive analyst. He appears to have studied entirely without assistance, and to have formed his taste and strengthened his powers by a diligent perusal of the continental models. In consequence, he was enabled to attack questions which none of his countrymen had entered upon…

The *Mathematical Essays*, edited by the young John Herschel, contains other interesting material, though none so impressive as *Logarithmic Transcendents.* The printed sheets were sent to Spence's widow in Scotland: unfortunately, copies became hard to find, and Spence's work was for long known to just a few. His place as perhaps the leading British analyst of his day is only now being appreciated. A fuller account is (Craik, 2013).

These three Scottish mathematicians, Knott, Sang and Spence, each influenced by Napier's logarithms in separate ways, deserve to be remembered: I am pleased to have had this opportunity to draw them to your attention.

References:

Craik, Alex D.D. 2003. The logarithmic tables of Edward Sang and his daughters. *Hist. Math.*, **30**, 47-84.

Craik, Alex D. D. 2007. Science and technology in 19th century Japan: The Scottish connection. *Fluid Dynamics Research* **39**, 24–48.

Craik, Alex D.D. 2013. Polylogarithms, functional equations and more: The elusive essays of William Spence (1777–1815). *Hist. Math.* **40**, 386-422.

Davidson, Andrew R. 1956. *The History of the Faculty of Actuaries in Scotland 1856–1956.* The Faculty of Actuaries, Edinburgh.

Horsburgh, Ellice M. 1914. *Napier Tercentenary Celebration. Handbook of the Exhibition of Napier Relics and of Books, Instruments, & Devices for Facilitating Calculation*. For The Royal Society of Edinburgh, Edinburgh.

Knott, Cargill .G. 1908. Dr Edward Sang's logarithmic, trigonometrical, and astronomical tables. *Proceedings of the Royal Society of Edinburgh* **28**, 183–196. Partly reproduced in (Horsburgh, 1914, 38–47).

Knott, Cargill G. (ed.), 1915. *Napier Tercentenary Memorial Volume*. Royal Society of Edinburgh and Longmans, Green & Co., London.

Peebles, D.Bruce 1897. "Edward Sang" (Obituary, with list of writings). *Proceedings of the Royal Society of Edinburgh* **21**, xvii–xxxii.

Sang, Edward 1871. *A new table of seven-place logarithms of all numbers from 20000 up to 200000.* Prepared under the auspices of the managers of the Associated Life Insurance Offices in Scotland. C. & E. Layton, London.(2nd edn: Elmslie Sang, Edinburgh, 1878. 2nd Edition improved: Williams & Norgate, London, 1883. 3rd Edition: Layton, London, 1915.)

Spence, William 1809. *Essay on the Theory of the Various Orders of Logarithmic Transcendents; with an inquiry into their applications to the integral calculus and the summation of series.* Murray, London and Constable, Edinburgh.

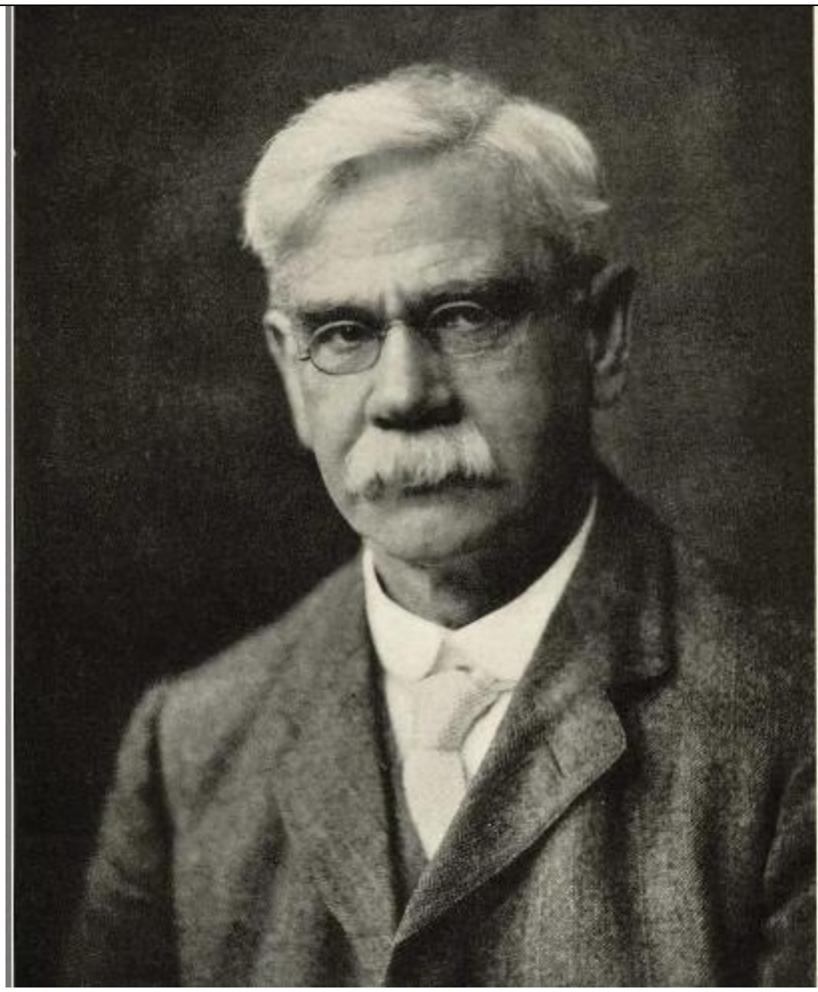
Spence, William 1814/1817. *Outlines of a Theory of Algebraic Equations deduced from the Principles of Harriott, and extended to the Fluxional and Differential Calculus.* Davis & Dickson for the author, London.

Spence, William 1819: *Mathematical Essays of the late William Spence, Esq….,* ed. J.F.W. Herschel, with biographical memoir by J. Galt. J. Moyes for T. and G. Underwood, London. Also 1820, for Oliver & Boyd, Edinburgh and G. & W.B. Whittaker, London.

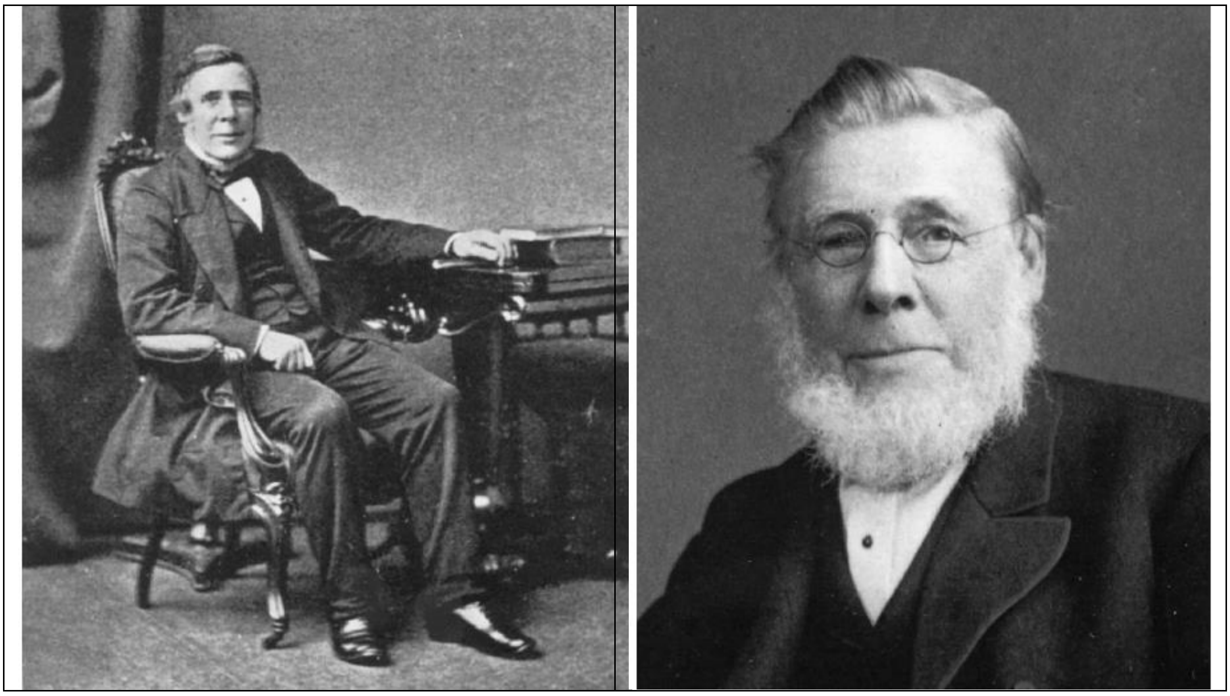
Whittaker, Edmund T. 1922–1923. "Cargill Gilston Knott, LL.D., F.R.S." Obituary notice. *Proc. Roy. Soc. Edinburgh* **43**, 236–248.

Figures (sent separately):

1. Photograph of Cargill Gilston Knott (reproduced from Whittaker, 1922-23).



2 (a) and (b). Photographs of Edward Sang.



3. Portrait of William Spence by an unknown artist. Watercolour and gouache on paper, 18 x 13**.** 8 cm. McLean Museum and Art Gallery, Greenock. Inverclyde Council No. 1978.373. Reproduced by permission.

[NOTE: Further illustrations of e.g. pages of Sang's tables and Spence title page can be added if desired.]