PROSTHAPHAERESIS[1]
-
The forerunner of the logarithm

$$\sin a \bullet \sin c = \frac{1}{2}\{\sin((90° - a) + c) - \sin((90° - c) - a)\}$$

(for a+c < 90°)

by
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[1] This is a shortened version of a paper published in 2012: see www.oughtred.org/jos/articles/PROSTHAPHAERESISandWERNERfinal.jmccLR8.8.pdf

Abstract

For most of his life Johannes Werner (1468-1522) was a priest and astronomer living in Nuremberg, Germany. He first published the prosthaphaeretic formulae (the term "prosthaphaeretic" coming from the Greek for addition and subtraction) around 1513 in a manuscript; this information is mainly supported by very intensive research carried out by Axel Anthon Björnbo (1874-1911) [Björnbo].

It is not exactly known if Werner was aware at that time of the advantageous use of the prosthaphaeretic formula for calculations with very large numbers; however, this can be assumed as being the case.

Moreover, strong evidence shows that neither the astronomer Tycho Brahe (1546-1601) nor his student Paul Wittich (1555?-1587) invented the prosthaphaeretic formula. However, Tycho Brahe was among the first, who - from 1580 to 1601 - took intensive advantage of the prosthaphaeretic formula for his astronomical calculations.

This paper reviews the historical background for the formulation and "re-invention" of prosthaphaeresis.

On the basis of the relevant literature it gives some practical examples as well as the mathematical-geometrical proof of the formula.

Introduction

For a long time, people have looked for ways to simplify computing procedures. It was not so important how difficult the calculations might have been; the goal was always to reduce the cost of computation, but without losing any accuracy.

Particularly in the field of astronomy, in which mathematics first developed, where computations with large numbers were (and still are) a necessity, the solutions were very expensive in time and effort. This particularly concerned the basic operations of arithmetic such as multiplication, so if it would be possible to simplify such operations, for example by reducing multiplication to addition, then that would be an ideal solution.

The most well-known example of this methodology would be logarithms, which were publicised in 1614 in Edinburgh by John Napier (1550-1617) in the first table of logarithms (Mirifici Logarithmorum Canonis Descriptio).

But what happened before then? How did astronomers do their calculations without knowledge of the logarithms?

The answer is that for about hundred years they used ***Prosthaphaeresis***, ((also written as Prosthaphärese, Prostaphärese or Prostaphairesis)

Literature on the subject of Prosthaphaeresis frequently mentions an incorrect name as its inventor; usually the discovery is attributed to the astronomer Tycho Brahe or to his pupil Paul Wittich, or sometimes even to Christopher Clavius. However, Brian Borchers gave a short overview of Prosthaphaeresis and its history in his article in the Journal of the Oughtred Society (JOS) [Borchers], and in that article he referred to its originator as being Johannes Werner.

Borchers' article stands as the starting point for this article, in which the background to the Prosthaphaeretic formula, and to Prosthaphaeresis itself, will be clarified from historical and mathematical viewpoints.

The term "Prosthaphaeresis" - meaning a system in which one uses addition and subtraction - also has other usages in astronomy; thus one speaks, for example, of prosthaphaeresis in connection with: aequinoctiorum; eccentrititatis; latitudinis; nodi pro eclipsius; orbis; tychonis; nodorum - "an orbiting body does not move itself evenly; it moves more slowly if the Sun is in the proximity of the body; faster, if the Sun moves away from it. " [Bialas]. However, these purely astronomical usages of the term "prosthaphaeresis" will not be considered further in this essay.

Johannes Werner (1468 - 1522) can be seen to be the discoverer of Prosthaphaeresis, and substantial support for this can be found in a work by Axel Anthon Björnbo [Björnbo]. As a pupil of the science historian Anton von Braunmühl, Björnbo took up von Braunmühl's references to some inconsistencies and went to Rome in 1901, so that he could read and study appropriate ancient material in the Vatican library.

Of particular interest, he found an undated manuscript, with the title: '"I. Joannis Verneri Norimbergensis "de triangulis sphaericis"' in four books, and also ' "II. Joannis Verneri Norimbergensis "de meteoroscopiis" ' in six books.

Queen Christina of Sweden had been in possession of this manuscript, probably between 1654 and 1689; this document had been previously owned by Jakob Christmann (1554 - 1613).

After Queen Christina's death in 1689, this manuscript (Codex Reginensis latinus 1259, i.e. Regina Sveciae Collection, item 1259) lay mainly un-noticed in the Vatican.

During further investigations it became clear that Werner was the editor and/or an author of the two handwritten parts, but that he did not physically write them himself.

As to the actual writer of the document, Björnbo identified a mathematically-inexperienced professional scribe of the time.[Björnbo; Pages 140, 141, 171].

The text of the first complete part of the manuscript (*de triangulis sphaericis*) can be found in Björnbo's work [Björnbo; Chapter 1] on pages 1 - 133. Later on Björnbo voices his opinions concerning this manuscript both in the "publisher remarks" [Björnbo; Chapter 3] and also in "text history" [Björnbo; Chapter 4] in a very detailed research report.

However, the extensive contents of that manuscript will not be dealt with further in this essay.

It should be mentioned however, that as an innovation for its time, the organisation/arrangement of the books concerning spherical triangles is to be seen [Björnbo; Chapter 1 and page 163]:

1. **An explanation of the different possible triangle forms (book I)**
    a. **A discussion concerning the spherical triangle**

2. **Solutions of the right-angled triangle (book II)**
    a. **The spherical-trigonometric basic formulae**
    b. **The solution of the right-angled spherical triangle**

3. **Solutions of the non-right-angled (obtuse) triangle (book III and IV)**
    a. **The solution of the obtuse triangle by decomposition into right-angled triangles (III)**
    b. **The solution of the obtuse spherical triangle by a prosthaphaeretically transformed Cosine rule (IV)**

The three categories stated are of the same structure as the contents of the *Opus Palatinum* of Rheticus of 1596, in so far as spherical triangles were concerned.

In chapter 5 [Björnbo; starting from page 177] is summarised the structure of the contents of the individual books in a tabular form.

Thus the findings of Björnbo, namely, the assumptions of Anton von Braunmühl [von Braunmühl 1897] which confirm the authority of the prosthaphaeretic formula, along with recent up-to-date work by David A. King [King] and Victor E. Thoren [Thoren] together constitute the foundation for the remainder of this article.

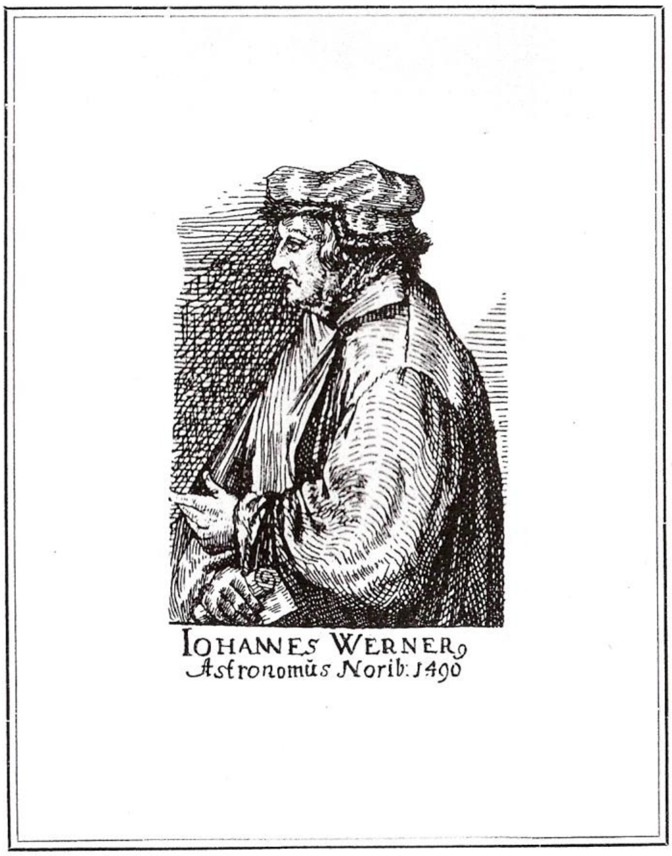
Historical

The history of Prosthaphaeresis is summarised in the time table in the appendix, and is derived from several literary sources [Björnbo; von Braunmühl].

Here now it is necessary to briefly describe the details of Johannes Werner's life, along with the steps in time of the development of Prosthaphaeresis, including its "rediscovery" and its sequence of publication.

Johannes Werner was born on 14 February 1468 in Nuremberg and died in (May?) 1522 in Nuremberg while he was in the post of parish priest in the municipality of St. Johannes.

In his spare time he worked as a mathematician, astronomer, astrologer, geographer and cartographer.



" Werner studied theology and mathematics in Ingolstadt from 1484 onwards. In 1490 he became a chaplain in Herzogenaurach.

From 1493 to 1497 he was in Rome.

In 1503 he was appointed as the vicar at the church in Wöhrd, a suburb of Nuremberg.

Afterwards he became a priest at the Johanniskirche in Nuremberg; he held this position up to his death.

Kaiser Maximilian I. appointed him the imperial Chaplain.

The IAU (International Astronomy Union; year unknown) honoured him by naming a crater on the Moon as "Werner".

Figure 5-1 Johannes Werner

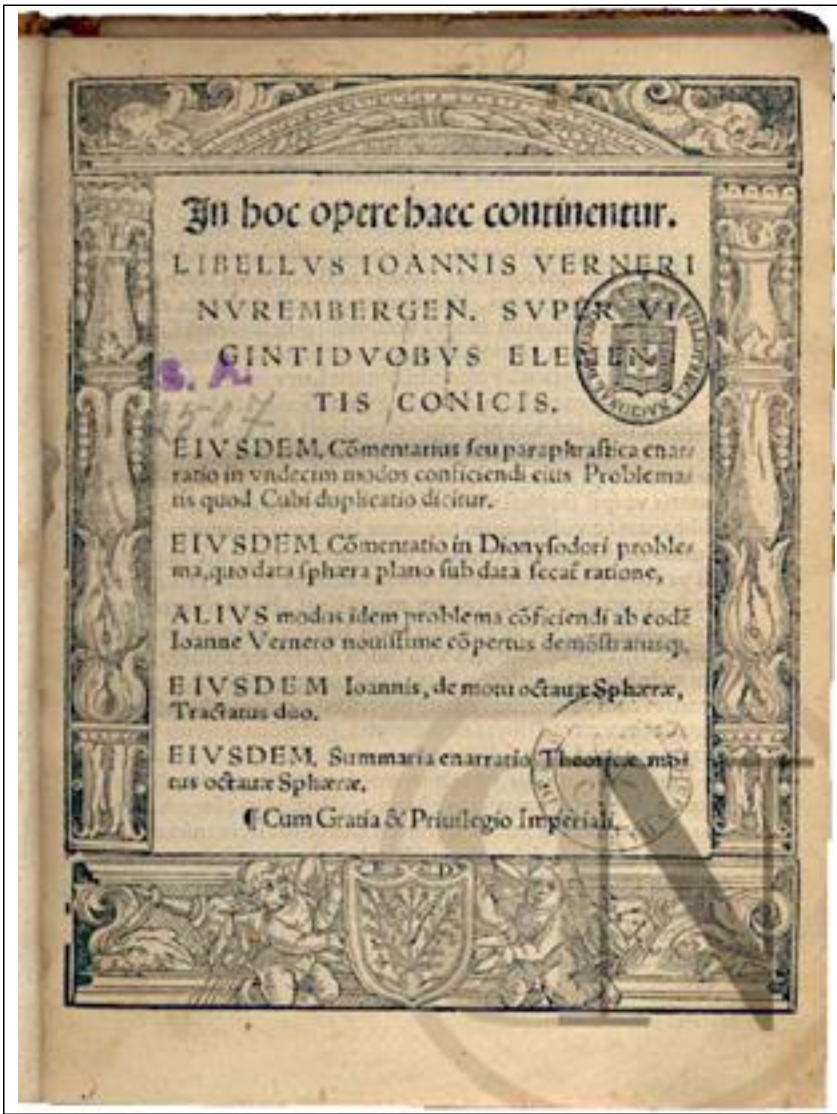
"Werner was very much interested in Astrology and created horoscopes for numerous well-known Nuremburg residents, including Erasmus Topler (1462-1512), Provost of St. Sebald, Willibald Pirckheimer (1470-1530), Christoph Scheurl II (1481-1542) and Sebald Schreyer (1446-1520). However, Werner gained harsh criticism from these activities. Lorenz Beheim (around 1457-1521), a choirmaster in Bamberg, wrote about him thus: "He always makes a big thing of his secrets, which however result in little honour for him. Mostly, if he wants to predict the truth, he invents it."

"Werner became friendly with Johannes Stabius (approximately 1460-1522). In co-operation with him, he developed numerous important works. Werner suggested the construction of a sun-dial, designed to show "Nuremberg time", which essentially meant that the clock should indicate the hours passed since sunrise. Stabius supplied a design, which Sebastian Sperantius (? - 1525) drew on the east choir of the Lorenzkirche in 1502.

Stabius also pushed Werner to publish his manuscripts. In November 1514, the compilation under the management of Conrad Heinfogel (? - 1517) left the printing press. Amongst other things therein, a certain form of the map projection is presented, which is known to historians as the Stabius-Werner Projection. In 1522 there appeared a second compilation (Fig. 5-2), which contained his work, "On the Motion of the Eighth Sphere" or "De motu octavae Sphaerae". He studied the precession of the stars from the geocentric point of view; however, for this he was fiercely criticised by Copernicus (1473 – 1543)."

This and other information, particularly also concerning Werner's meteorological activities, can be found on the Internet under [Nuremberg] and [Wikipedia Werner].

A first compilation was published in 1514 under the title: "In hoc opere haec continentur: Nova translatio primi libri geographiae Cl. Ptolemaei, quae quidem translatio verbum habet e verbo fideliter expressum Ioanne Vernero Nurembergensi interprete……", containing work by him and by other authors.



"From that compilation and from his own publications, what we know of Werner's life is only the following:

Starting before 1513, but probably after 1505, Werner wrote five books concerning spherical triangles with the title "Liber de triangulis sphaericis" or "Liber sphaeraliurn triangulorum".

During the years 1514 to 1522 this work underwent editing and collation.

Werner was very eager to have the work published, particularly because he was very aware as early as 1514 that the prosthaphaeretic method had a great value. " [Björnbo, P. 157]

Figure 5-2 Compilation of 1522 [according to Mehl - from the library in Lisbon]

Björnbo draws this conclusion from the similarity of the contents of the manuscript with the contents of the compilations. On the one hand it concerned thereby the amazing similarity and/or sameness of the solution of a triangle using orthogonal projection. Therein Werner had "already written in a pure form the prosthaphaeretic method and its application for the practical transformation of the Cosine Rule, i.e. the second main rule of spherical Trigonometry*."

*

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c}$$

Further he says [Björnbo; Page 155]: "In the compilation of the year 1522 there appears in Werner's book "De motu octavae Sphaerae"…. in the triangle (star [2]; pole of the ecliptic; north pole) the height of the Star (λ) by its width (ß), its declination (δ) and its inclination to

[2] The triangle is determined by the three corners: star; pole of the ecliptic; north pole

the ecliptic (ε), i.e. that the angle of a skew spherical triangle is numerically determined by its three given sides…."

The fact that the emergence of Prosthaphaeresis must have taken place after 1505 Björnbo takes from one of the available quotations in the translation of Euclid's work by Zamberti (Bartholomäo Zamberto Veneto) which became available only after 1505.

After it was clear that the manuscript Cod. Reg. 1259 had its origin in these two works by Johannes Werner, the search was on, after Werner's death, to find out the development and the whereabouts of this Cod. Reg. 1259.

Up to the death of Werner in the year 1522 the two works had still not been printed, or at least, no appropriate references or copies have been found from that time.

The contents of Book I - Joannis Verneri "*Norimbergensis de triangulis sphaericis*" in four books, as well as Book II - Joannis Verneri "*Norimbergensis de meteoroscopiis*" in six books, were however well known to Werner's contemporaries, including Johann Wilhelm von Loubemberg and his colleague Peter Apian.

The bibliographer Konrad Gesner (1516-1576) describes in 1555 that the Nuremberg mathematician and mechanic George Hartmann (1489-1564) saved the two works of Werner from destruction. According to Doppelmayr, Hartmann probably handed over these and other works from Werner's estate in 1542 in Nuremberg to George Joachim Rheticus (1514-1576; who lived from 1554 in Kraków as a practicing physician).

G. Eneström [Eneström] determined that both works of Johannes Werner were published by Rheticus in the year 1557 in Kraków.

However this publication contained, apart from the title page, only the ten-page introduction (The Prooemium) by Rheticus and nothing else which Werner wrote.

The title page contains clear references to the titles of the two books mentioned above.

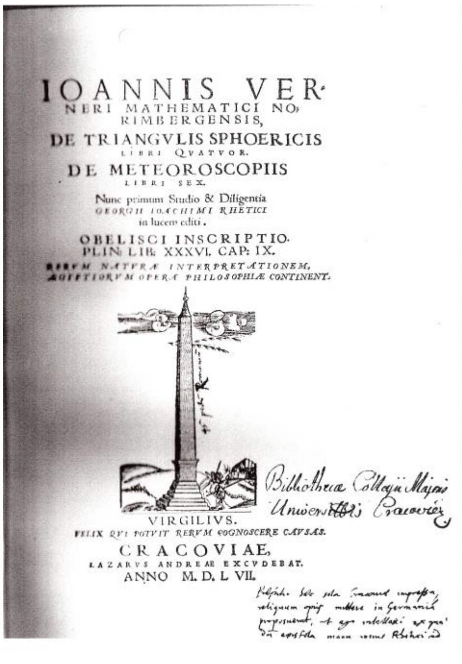


Figure 5-3 The Title page of the Kraków publication.

Björnbo sees as an explanation for the absence of the text, the fact that Rheticus, and after his death his pupil Valentinus Otho (approx. 1550-1605), had incorporated both the arrangement (a systematic presentation of different triangle forms) and the contents of the "De triangulis sphaericis", in the great book of tables *Opus Palatinum* (which was published in 1596) and they had perfected the philosophies of Johannes Werner, whom they both admired and respected.

However Rheticus did not himself support the solution of spherical triangles either by Ptolemaios' method or by Geber's method (which was developed by Peuerbach, Regiomontanus and Werner).

So, he developed his own method independently; this method derived from the geometry of pyramids, using common points at the centre of the sphere; this latter methodology is derived from Copernicus [Björnbo, page 163 foot-note 2].

Rheticus was the only pupil of Copernicus and by his publication of the famous "De revolutionibus orbium coelestium Libri VI" had himself taken up the cause of providing a reliable sine table.

Thus Björnbo assumes the manuscript Cod Reg 1259 lying in the Vatican was in the Rheticus' possession and represented a copy of the original, and that it should serve as a beginning point.

This printed manuscript - which contained no drawings - fell into the hands of his pupil Valentinus Otho after the death of Rheticus in 1576.

From this bequest the manuscript went to the Heidelberger professor Jakob Christmann (1554-1613 {Björnbo P. 165 incorrectly describes the date of death as 1630}), who quoted from the two works of Werner in his book *"Theoria lunae"* (1611), and even indicated that he possessed the two books.

In his dissertation of 1924, Erwin Christmann (a later successor of Jakob Christmann) wrote the following [Christmann]:

- *" The "Theoria lunae" plays a remarkable role in the history of trigonometry, as it gave in an appendix, information concerning the inventor of the prosthaphaeretic method.*

  Until the discovery of Werner's two documents, "de triangulis sphaericis" and "de meteoroscopiis" by A. Björnbo in 1902 in the Vatican library in Rome, the "Theoria lunae" was one of the few sources to bring clarity over this long disputed question.

  For von Braunmühl in 1899 in his "Lectures on the History of Trigonometry", Christmann's writing is the most outstanding support for his proof of the invention of the prosthaphaeretic method by Johannes Werner.

  Christmann explained here, that the manuscript of that work was well-known to him, -although it is not known whether it was the original manuscript, later lost, or the printed copy from the Vatican library, which was available to him -. Werner developed and in figures described therein the Prosthaphaeresis. He defended this against Tycho Brahe, who with his pupil Wittich, were generally regarded as being the inventors. Christmann is probably referring to a transcript, which would be good as a basis on which to work; his words therefore do not suggest a deliberate deception.

- *"Even today these relationships are not as clear as could be desired. It is feasible to recognise Werner as the inventor of the method and as the person who saw the opportunities for its possible use; however, in reality he is more the re-discoverer of the prosthaphaeretic formulae, as they were already well known to Arabic mathematicians. On the other hand, one must be objective and the trustworthy mathematical and astronomical circle of Count Wilhelm of Hessen above all ascribe to Wittich and Tycho Brahe the exclusive merit of the general introduction of the use of the prosthaphaeretic formulae in calculation. The meaning of their activity must be recognised all the more, in that the holy-of-holies inventors of logarithms and of their practical use had not become available. Furthermore, that this was not a collection of formulae by Wittich and Tycho Brahe can be proven by comparative research.*

- *In addition to the information given in the "theoria lunae" Christmann brings a full development of the method and key phrases from the triangle theory, so far as it required.*
  He had already summarised these into his works "observationum solarium libri tres, in quibus explicatur versus motus Solis in sodiaca et universa doctrina triangulorum ad rationes apparentius coelestium accomodatur Basel 1601". In another work called "nodus Cordinis ex doctrina sinum explicatus 1612" he taught the solution of geometrical problems with the help of sines, instead of using algebraic methods.

- *Although today, by the rediscovery of Werner's trigonometrical work, the "theoria lunae" with its data has receded into the background, nevertheless its existence remains historically notable, particularly because its statements were, as a result of recent investigations, accepted as correct and also because together with the two writings from the years 1601 and 1612 written by a professor from Heidelberg interested in trigonometry, it puts down a clear testimony."*

Anton von Braunmühl based his remarks for the development of the Prosthaphaeresis particularly on the statements of Jakob Christmann. He sees the origin of these formulae as being with Ibn Yunus, an Arab mathematician who died in 1009. However, according to David A. King [King], on the basis of new knowledge which he acquired while working on his thesis, this idea is no longer valid.

What role does Tycho Brahe (1546-1601), the Danish astronomer, play in connection with Prosthaphaeresis, which he himself began to use in 1580?

According to [von Braunmühl 1899] "*Tycho Brahe knew the source, in which Werner, using his trigonometrical books, applies the prosthaphaeretic method in order to find the elevation of Spica Virginis, because he often speaks of Werner´s writing "De motu octavae Sphaerae" and he (Tycho) particularly drew upon this observation of Spica. However the wording of that source could make it attentive only on the existence of a more practical calculation procedure, than the usual one is, the procedure itself was absolutely not to be taken out of that source."*

It is possible that Brahe had direct access to Johannes Werner's manuscript, or it can surely be assumed that the manuscript's contents were known to him. [Björnbo, Page 168 ff] There are several ways in which this might have happened; see also [Thoren]:

1. During Brahe's visits to Wittenberg in the years 1566 or 1568-1569 or 1575, he may have seen Johannes Werner's books about triangles.

2. Paul Wittich and Brahe could have developed their own prosthaphaeretic method in 1580.

3. Reimarus Ursus (Nicolai Reimers; 1551-1600) - during a visit to the island Hven, where Brahe worked in 1584, may have stolen the prosthaphaeretic formula, and was thereafter considered as an intimate enemy of Brahe. In Ursus` *Fundamentum Astronomicum* (Strasbourg 1588) Johannes Werner's prosthaphaeretic formula is published for the first time.

4. Jost Bürgi, who was in contact with Wittich, may have played a role and may have received the formula from Bürgi in Kassel - according to [Thoren] and [Lutstorf], Bürgi may then have provided the geometrical proofs.

5. Johann Richter (also known as Praetorius) (1537-1616) saw the book concerning spheres in 1569 written by Rheticus (he writes about it in 1599) and was from 1571-1576 a Professor of Mathematics in Wittenberg. According to a letter which Brahe wrote in 1588 to Hayck, he had not met Praetorius in 1575.

6. The role of Paul Wittich - to whom Brahe in 1592 (5 years after Wittich's death) ascribed the discovery of the Prosthaphaeresis. This is also proposed by [Thoren], who differentiates between the prosthaphaeretic formula itself, and actual computations with that formula.

Possibly it was a mixture of the above points, which led to the fact that Tycho Brahe became acquainted with Prosthaphaeresis and then further developed it with Paul Wittich and learned how to use it. Anton von Braunmühl [von Braunmühl; Part 1, page 193] speaks therefore also of a "re-invention" of Prosthaphaeresis by Brahe in the year 1580. *Also Kepler (wann ??? nicht ermittelbar) refers to Prosthaphaeresis on one occasion as "Artificium Tychonicum", then again as "Negotium Wittichianum" and finally as "Regula Wittichiana"* [von Braunmühl 1899].

The historical journey of the manuscript and of the formulae are graphically summarised in the appendix.

Now, the significance of Regiomontanus (Johannes Mueller, born in 1436 in Königsberg near Hassfurt - died in 1476 in Rome) concerning the work of Johannes Werner, will be considered. Björnbo [Björnbo, page 172ff] explains the fact that Werner gained access to Regiomontanus' works, among other things the 5 "unfinished and mutilated" triangle books quite late - in fact, as late as 1504. Werner was not happy about this, and perhaps for this reason makes no reference in his own work to Regiomontanus, and does not cite the latter's work. Perhaps in addition, because he was very familiar with the works of Euclid, Menelaus, Geber, Ptolemaios and von Peurbachs as used by Regiomontanus, he did not want to repeat Regiomontanus' work. However similarities can be seen in the ideas and in some of the expressions found in Regiomontanus' work and in Werner's work.

There frequently occur in connection with the history of Prosthaphaeresis names of some very well-known and of some less well-known scholars, who cannot be dealt with in great detail here, but who should not be completely ignored. Their roles and work in connection with Prosthaphaeresis are probably worth a completely separate investigation, but their names and some details are given here:

In the first place Jost Bürgi (1552 - 1632)

- Peter Apian (1495 - 1552)
- Erasmus Reinhold (1511 - 1553)
- Bartholomäus Scultetus / Schulz (1532 - 1614)

- Christoph Clavius (1537 - 1602); (1538 - 1612, is also mentioned as the inventor of the Prosthaphaeresis [Symposium 2005]) – he is not, however.
- Nicolaus Reimers /Reimarus Ursus (1551 - 1600)
- Paul Wittich (1555 - 1587)
- Melchior Jöstel (1559 - 1611) and his handwritten treatise *"Logistica Prosthaphaeresis Astronomica"* which can be found in the library of the Austrian National Library, Vienna (Cod. palat. 10686-27) [von Braunmühl 1899] as well as in the Dresden Landesbibliothek [Folkerts].
- Christian Severin Longomontan (1562 - 1647)

and finally Ibn Yunus (around 1000).

Apart from the first and last, the above names are chronologically ordered according their years of birth.

Much introductory information and references to these above people can be found in [von Braunmühl 1900], [Lutstorf] and [Thoren] and also in [Gingerich 1988] and [Gingerich 2005].

Mathematical

To remind the reader, Prosthaphaeresis provides a methodology by the means of which the process of multiplication can be converted into an addition or a subtraction by the use of trigonometric formulae. This technique provided a substantial easing of work for the astronomers of the time.

Looking at books of formulae or on the Internet [Mathworld 11 and 12] it becomes clear that there are many ways of expressing the Prosthaphaeresis formulae. These formulae, which we know as the "prosthaphaeretic formulae" or as the "prosthaphaeresis formulae", are also known as the "Werner Formulae" or as the "Werner Formulas" (Fig. 6-1).

Geometry > Trigonometry > Trigonometric Identities

Werner Formulas

The Werner formulas are the trigonometric product formulas

$$2\sin\alpha\cos\beta = \sin(\alpha-\beta)+\sin(\alpha+\beta) \quad (1)$$
$$2\cos\alpha\cos\beta = \cos(\alpha-\beta)+\cos(\alpha+\beta) \quad (2)$$
$$2\cos\alpha\sin\beta = \sin(\alpha+\beta)-\sin(\alpha-\beta) \quad (3)$$
$$2\sin\alpha\sin\beta = \cos(\alpha-\beta)-\cos(\alpha+\beta). \quad (4)$$

This form of trigonometric functions can be obtained in *Mathematica* using the command TrigReduce[*expr*].

SEE ALSO: Prosthaphaeresis Formulas. [Pages Linking Here]

LAST MODIFIED: February 17, 2006

CITE THIS AS:

Weisstein, Eric W. "Werner Formulas." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/WernerFormulas.html

Figure 6-1 The Werner Formulae

The URL on the above website leads to the Prosthaphaeresis Formulae as shown below in Fig. 6-2 and which are known as "Simpson's Formulae" or "Simpson's Formulas". However these formulae differ in their representation and in their ease of use.

Geometry > Trigonometry > Trigonometric Identities

Prosthaphaeresis Formulas

The Prosthaphaeresis formulas, also known as Simpson's formulas, are trigonometry formulas that convert a product of functions into a sum or difference. They are given by

$$\sin\alpha + \sin\beta = 2\sin\left[\tfrac{1}{2}(\alpha+\beta)\right]\cos\left[\tfrac{1}{2}(\alpha-\beta)\right] \quad (1)$$

$$\sin\alpha - \sin\beta = 2\cos\left[\tfrac{1}{2}(\alpha+\beta)\right]\sin\left[\tfrac{1}{2}(\alpha-\beta)\right] \quad (2)$$

$$\cos\alpha + \cos\beta = 2\cos\left[\tfrac{1}{2}(\alpha+\beta)\right]\cos\left[\tfrac{1}{2}(\alpha-\beta)\right] \quad (3)$$

$$\cos\alpha - \cos\beta = -2\sin\left[\tfrac{1}{2}(\alpha+\beta)\right]\sin\left[\tfrac{1}{2}(\alpha-\beta)\right]. \quad (4)$$

This form of trigonometric functions can be obtained in *Mathematica* using the command `TrigFactor`[*expr*].

Figure 6-2 Prosthaphaeresis Formulae from Mathworld.

(In German linguistic usage [von Braunmühl 1900] these formulae shown in Fig. 6-1 are known as "Die prosthaphäretischen Formeln".)

In more modern collections of formulae these names are not used, but instead the formulae are referred to as "products of trigonometric functions" [Bartsch] - see Fig. 6-3.

Products of trigonometric functions

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta) + \cos(\alpha+\beta)]$$

$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta) + \sin(\alpha-\beta)]$$

$$\cos\alpha\sin\beta = \frac{1}{2}[\sin(\alpha+\beta) - \sin(\alpha-\beta)]$$

Figure 6-3 The Prosthaphaeretic formulae as "products of trigonometric functions".

Applications

As a first application, a way to do multiplication is shown with the help of Formula 1:

$$A \bullet B = \sin a \bullet \cos b = \tfrac{1}{2}\,[\sin (a + b) + \sin (a - b)] \quad \textbf{(Formula 1)}$$

with the factors A = 0.61566 and B = 0.93969 [Enzykl, page 70]. From the table in Fig 7-1 we read for factor A an angle of a = 38° in the sine column (green), and for B an angle of b = 20° from the cosine column (red) (see red ellipses).

Va. NATÜRLICHE WERTE FÜR SINUS, TANGENS, KOSINUS, KOTANGENS

Grad	sin	tan	
0	0,0000	0,0000	90
1	0175	0175	89
2	0349	0349	88
3	0523	0524	87
4	0698	0699	86
5	0872	0875	85
6	1045	1051	84
7	1219	1228	83
8	1392	1405	82
9	1564	1584	81
10	0,1736	0,1763	80
11	1908	1944	79
12	2079	2126	78
13	2250	2309	77
14	2419	2493	76
15	2588	2679	75
[illegible]	[illegible]	2867	74
17	2924	3057	73
18	3090	3249	72
[illegible]	[illegible]	3443	71
20	0,3420	0,3640	70
21	3584	3839	69
22	3746	4040	68
23	3907	4245	67
24	4067	4452	66
25	4226	4663	65
26	4384	4877	64
27	4540	5095	63
28	4695	5317	62
29	4848	5543	61
30	0,5000	0,5774	60
31	5150	6009	59
32	5299	6249	58
33	5446	6494	57
34	5592	6745	56
35	5736	7002	55
[illegible]	[illegible]	7265	54
37	6018	7536	53
38	6157	7813	52
39	6293	8098	51
40	0,6428	0,8391	50
41	6561	8693	49
42	6691	9004	48
43	6820	9325	47
44	6947	9657	46
45	7071	1,0000	45
	cos	cot	Grad

Grad	sin	tan	
45	0,7071	1,000	45
46	7193	036	44
47	7314	072	43
48	7431	111	42
49	7547	150	41
50	0,7660	1,192	40
51	7771	235	39
52	7880	280	38
53	7986	327	37
54	8090	376	36
55	8192	428	35
[illegible]	[illegible]	483	34
57	8387	540	33
58	8480	600	32
[illegible]	[illegible]	664	31
60	0,8660	1,732	30
61	8746	804	29
62	8829	881	28
63	8910	963	27
64	8988	2,050	26
65	9063	145	25
66	9135	246	24
67	9205	356	23
68	9272	475	22
69	[illegible]	[illegible]	[illegible]
70	0,9397	2,747	20
71	[illegible]	[illegible]	[illegible]
72	9511	3,078	18
73	9563	271	17
74	9613	487	16
75	9659	732	15
76	9703	4,011	14
77	9744	331	13
78	9781	705	12
79	9816	5,145	11
80	0,9848	5,671	10
81	9877	6,314	9
82	9903	7,115	8
83	9925	8,144	7
84	9945	9,514	6
85	9962	11,43	5
86	9976	14,30	4
87	9986	19,08	3
88	9994	28,64	2
89	9998	57,29	1
90	1,0000	∞	0
	cos	cot	Grad

That is, for the two figures in the blue boxes we see the results for a + b = 58° and for a - b = 18°.

With sin 58° = 0.8480 and sin 18° = 0.3090 their sum is 1.1570.

Halving this gives 0.5785, which is the correct product of 0.61566 • 0.93969 to 4 places (to be more exact: 0.578529545).

The absolute error between 0.578500000 and 0.578529545 is 0.000029545, which corresponds to a relative error of 0.0051%.It is easy to recognise that using a sine table with higher precision, will result in a higher accuracy.

With Regiomontanus' seven-place values from his sine tables of 1541 we find sin 18° = 0.3090170 and sin 58° = 0.8480481, giving a sum of 1.1570651 and a half thereof of 0.578532550 which is a value much closer to the exact result with an absolute error = 0.000003005.

Figure 7-1 four-digit table from [Enzykl, page 805]

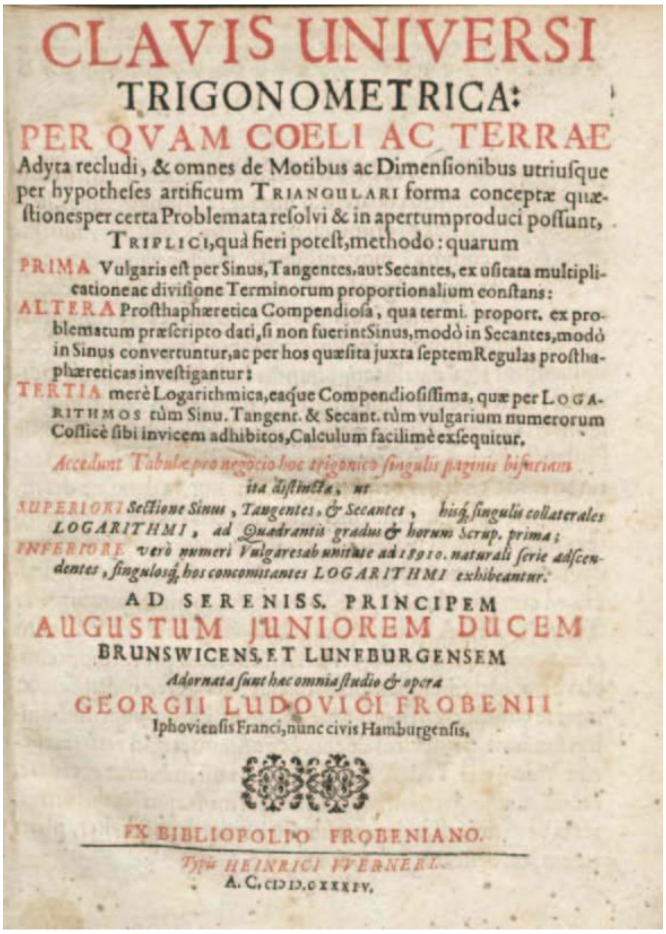
It is also possible to do division is this manner. Replacing cos b by (1/sec b) and thereby getting another angle for b, it can be further calculated with the right side of the formula 1.

As the connoisseur can imagine, it is also possible to reverse this process to, for example, calculate an addition by using a multiplication. This method is very geeky, but might be of theoretical interest to a slide rule user!

George Ludwig FROBENIUS[3] (25.8.1566 in Iphofen - 21.7.1645 in Hamburg) was a Polyhistor (Universal scholar), a mathematician, a bookseller and a Hamburg publisher.

He lived in a time of change in computing methods as used in astronomical applications.

These circumstances were shown in his ***Clavis Universi Trigonometrica*** [Frobenius] in which arithmetical examples of the known methods of calculation were presented.


CLAVIS UNIVERSI
TRIGONOMETRICA:
PER QVAM COELI AC TERRAE
Adyta recludi, & omnes de Motibus ac Dimenſionibus utriuſque per hypotheſes artificum TRIANGULARI forma conceptæ quæſtiones per certa Problemata reſolvi & in apertum produci poſſunt, TRIPLICI, quà fieri poteſt, methodo: quarum
PRIMA Vulgaris eſt per Sinus, Tangentes, aut Secantes, ex uſitata multiplicatione ac diviſione Terminorum proportionalium conſtans:
ALTERA Proſthaphæretica Compendioſa, qua termi. proport. ex problematum præſcripto dati, ſi non fuerint Sinus, modò in Secantes, modò in Sinus convertuntur, ac per hos quæſita juxta ſeptem Regulas proſthaphæreticas inveſtigantur:
TERTIA merè Logarithmica, eaque Compendioſiſſima, quæ per LOGARITHMOS tùm Sinu. Tangent. & Secant. tùm vulgarium numerorum Coſſicè ſibi invicem adhibitos, Calculum facilimè exſequitur.
Accedunt Tabulæ pro negocio hoc trigonico ſingulis paginis bifariam ita diſtinctæ, ut
SUPERIORI Sectione Sinus, Tangentes, & Secantes, hiſq; ſingulis collaterales LOGARITHMI, ad Quadrantis gradus & horum Scrup. prima;
INFERIORE verò numeri Vulgares ab unitate ad 18910. naturali ſerie adſcendentes, ſingulosq; hos concomitantes LOGARITHMI exhibeantur.
AD SERENISS. PRINCIPEM
AUGUSTUM JUNIOREM DUCEM
BRUNSWICENS. ET LUNEBURGENSEM
Adornata ſunt hæc omnia ſtudio & opera
GEORGII LUDOVICI FROBENII
Iphovienſis Franci, nunc civis Hamburgenſis.

EX BIBLIOPOLIO FROBENIANO.
Typis HEINRICI WERNERI.
A. C. CIↃ IↃ.C XXXIV.

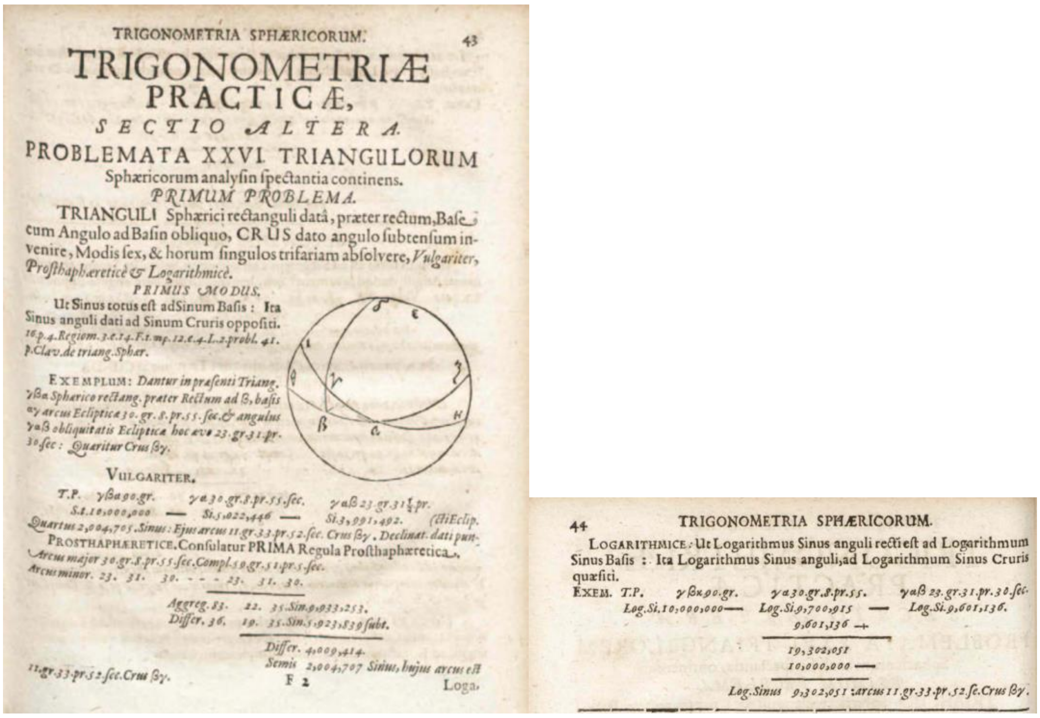
Figure 7-2 2nd title page Frobenius

Frobenius used three methods, which he named

- "Prima (1st)" or "Vulgaris",
- "Altera (2nd)" or "Prosthaphaeretice"
- "Tertia (3rd)" or "Logarithmice".

In the following examples, the three different methods are demonstrated and described to demonstrate the computation of an elevation, which results from the cutting across two diameters (great circles around a sphere).The two diameters are taken from astronomy (spherical trigonometry) and represent the equator and the diameter of the Earth in line with the ecliptic.

3 G.L. Frobenius is not the originator of the "Satz des Frobenius", who is Ferdinand Georg Frobenius, Mathematician (1849 - 1917)



TRIGONOMETRIA SPHÆRICORUM. 43

TRIGONOMETRIÆ PRACTICÆ,
SECTIO ALTERA.
PROBLEMATA XXVI. TRIANGULORUM Sphæricorum analysin spectantia continens.
PRIMUM PROBLEMA.

TRIANGULI Sphærici rectanguli datâ, præter rectum, Base cum Angulo ad Basin obliquo, CRUS dato angulo subtensum invenire, Modis sex, & horum singulos trifariam absolvere, *Vulgariter, Prosthaphæreticè & Logarithmicè.*

PRIMUS MODUS.

Ut Sinus totus est ad Sinum Basis : Ita Sinus anguli dati ad Sinum Cruris oppositi. *16.p.4.Regiom.3.e.14.F.1.mp.12.e.4.L.2.probl.41. p.Clav.de triang.Sphær.*

EXEMPLUM: *Dantur in præsenti Triang. γβα Spharico rectang. præter Rectum ad β, basis αγ arcus Ecliptica 30.gr.8.pr.55.sec.& angulus γαβ obliquitatis Ecliptica hoc ævo 23.gr.31.pr.30.sec: Quæritur Crus βγ.*

VULGARITER.

T.P. γβα 90.gr. — γα 30.gr.8.pr.55.sec. — γαβ 23.gr.31½.pr.
S.t.10,000,000 — Si.5,022,446 — Si.3,991,492. (ctiEclip.
Quartus 2,004,705.Sinus: Ejus arcus 11.gr.33.pr.52.sec. Crus βγ. Declinat. dati pun-

PROSTHAPHÆRETICE. Consulatur PRIMA Regula Prosthaphæretica.
Arcus major 30.gr.8.pr.55.sec.Compl.59.gr.51.pr.5.sec.
Arcus minor. 23. 31. 30. - - - 23. 31. 30.

Aggreg. 83. 22. 35.Sin.9,933,253.
Differ. 36. 19. 35.Sin.5,923,839 subt.

Differ. 4,009,414.
Semis 2,004,707 Sinus, hujus arcus est 11.gr.33.pr.52.sec.Crus βγ.

F 2 Loga.

44 TRIGONOMETRIA SPHÆRICORUM.

LOGARITHMICE: Ut Logarithmus Sinus anguli recti est ad Logarithmum Sinus Basis : Ita Logarithmus Sinus anguli, ad Logarithmum Sinus Cruris quæsiti.

EXEM. *T.P. γβα 90.gr. — γα 30.gr.8.pr.55. — γαβ 23.gr.31.pr.30.sec.*
Log.Si.10,000,000 — Log.Si.9,700,915 — Log.Si.9,601,136.
9,601,136 +
19,302,051
10,000,000 —
Log.Sinus 9,302,051 : arcus 11.gr.33.pr.52.sec.Crus βγ.

Figure 7-3 Example pages from Frobenius for the computation of a side in a spherical triangle

In the spherical triangle **αβγ**, the angle **a** opposite the side **βγ** is to be computed.

Two angles and the side opposite the other angle (**ß**) are given.

The characteristic (and simplification) is that this concerns a right-angled spherical triangle; the right angle (90°) is at the angle **ß**.

For the computation of the side **βγ**, the sine rule is applicable; within a right-angled spherical triangle this is expressed as:

Sine rule: sin a = sin α • sinγα / sin ß

The angle **α** has a value of 23 degrees, 31 minutes and 30 seconds - and corresponds thereby to the angle of the ecliptic. The side **γα** has a value of 30 degrees, 8 minutes and 55 Seconds. Because
sin 90° = 1, the formula simplifies to:

sin a = sin α • sin γα

Following the first "Vulgariter" method, there results the calculation process represented in table 7-1:

Table 7-1 Calculation method using multiplication of sines

1. Vulgariter	Angle/Side	Trig Funct	Degree	Pr	Sec	
	α		23	31	30	
	γα = b		30	8	55	
	β		90			
	Si α	3.991.492				
		times				
	Si γα	5.022.446				
		divided by				
	S.t. (sinus totus = β)	10.000.000				
		equals				
	Quartus = si α x si γα	2.004.705				
Solution:	Sinus: arcus ejus		11	33	52	Crus βγ

According to this method the seven-place sines of the angles were determined and **multiplied** with one another.

The resultant solution for the side **ßγ = a**: 11 degrees, 33 minutes 52 seconds.

The 2nd method is "Prosthaphaeretic", which works according to the following formula:

$$\sin \gamma a \bullet \sin \alpha = \frac{1}{2}\{\sin ((90° - \gamma a) + \alpha) - \sin ((90° - \alpha) - \gamma a)\}$$

Table 7-2 Calculation method using Prosthaphaeresis

2. Prosthaphäretice	Angle/Side	Degree	Pr	Sec			Degree	Pr	Sec		Sinus	Arcus		
γα	Arcus Major	30	8	55		Compl γα	59	51	5			Grad	Pr	Sec
α	Arcus Minor	23	31	30		α	23	31	30					
					plus	Aggreg.	83	22	35		9,933,253			
										minus				
					minus	Differ.	36	19	35		5,923,843			
										Differ.	4,009,410			
										divided by 2				
Solution:						Crus βγ				Semis	2,004,705	11	33	52

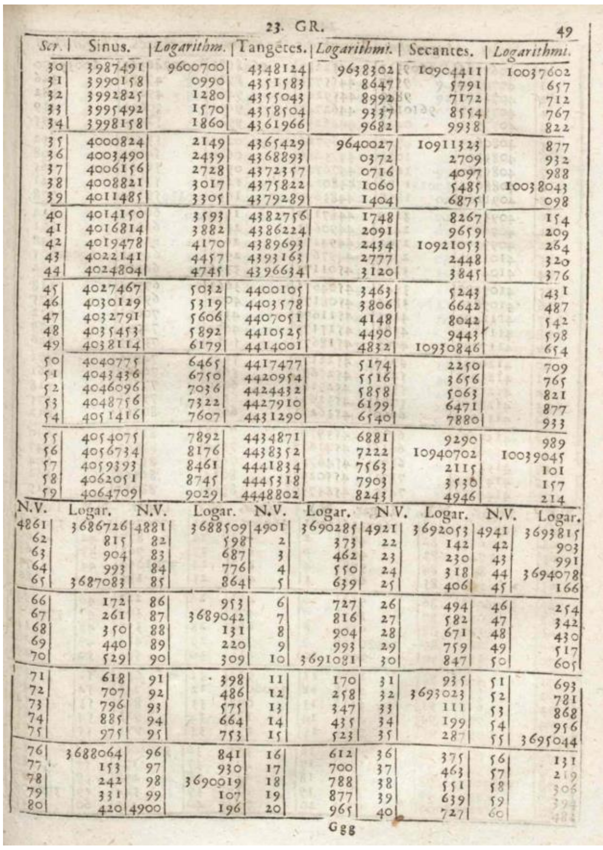
In table 7-2 the calculation method is shown, in which the result is calculated using prosthaphaeretic formula. To remind the reader: the product to be computed (sin **γα** times sin **α**) can be computed by the **Addition and Subtraction** of Sines. Only at the conclusion there is an additional, simple, division (Semis) by 2. We see here a somewhat elaborate calculation process like the above Vulgariter method; however simpler calculation steps are used.

The simplest and fastest way for the computation of the height is the logarithmic method shown in table 7-3. In addition it was necessary, to look up the logarithms associated with the sines and **add and/or subtract** these. The use of suitable tables was trusted by astronomers of that time, because there already existed appropriate tables for the trigonometric functions and for their logarithms.

Table 7-3 Calculation method using Logarithms

3. Logarithmice	Angle/Side	Degree	Pr	Sec		Log sin ∢		Log Sinus	Degree	Pr	Sec
	α	23	31	30		9,601,136					
	γα	30	8	55	plus	9,700,915					
						19,302,051					
	β	90			minus	10,000,000					
Solution:						9,302,051	Crus βγ	**9,302,051**	**11**	**33**	**52**

Frobenius had included a detailed table (see Fig. 7-4) in his comprehensive *Clavis Universi Trigonometrica* (323 pages text book plus 184 pages of tables); in this table, sines as well as tangents and secants, and their logarithms, for each minute of angle are exactly set down. Additionally, the Briggs logarithms of the numbers are tabulated.



23. GR. 49

Scr.	Sinus.	Logarithm.	Tangetes.	Logarithmi.	Secantes.	Logarithmi.
30	3987491	9600700	4348124	9638302	10904411	10037602
31	3990158	0990	4351583	8647	5791	657
32	3992825	1280	4355043	8992	7172	712
33	3995492	1570	4358504	9337	8554	767
34	3998158	1860	4361966	9682	9938	822
35	4000824	2149	4365429	9640027	10911323	877
36	4003490	2439	4368893	0372	2709	932
37	4006156	2728	4372357	0716	4097	988
38	4008821	3017	4375822	1060	5485	10038043
39	4011485	3305	4379289	1404	6875	098
40	4014150	3593	4382756	1748	8267	154
41	4016814	3882	4386224	2091	9659	209
42	4019478	4170	4389693	2434	10921053	264
43	4022141	4457	4393163	2777	2448	320
44	4024804	4745	4396634	3120	3845	376
45	4027467	5032	4400105	3463	5243	431
46	4030129	5319	4403578	3806	6642	487
47	4032791	5606	4407051	4148	8042	542
48	4035453	5892	4410525	4490	9443	598
49	4038114	6179	4414001	4832	10930846	654
50	4040775	6465	4417477	5174	2250	709
51	4043436	6750	4420954	5516	3656	765
52	4046096	7036	4424432	5858	5063	821
53	4048756	7322	4427910	6199	6471	877
54	4051416	7607	4431290	6540	7880	933
55	4054075	7892	4434871	6881	9290	989
56	4056734	8176	4438352	7222	10940702	10039045
57	4059393	8461	4441834	7563	2115	101
58	4062051	8745	4445318	7903	3530	157
59	4064709	9029	4448802	8243	4946	214

N.V.	Logar.	N.V.	Logar.	N.V.	Logar.	N.V.	Logar.	N.V.	Logar.
4861	3686726	4881	3688509	4901	3690285	4921	3692053	4941	3693815
62	815	82	598	2	373	22	142	42	903
63	904	83	687	3	462	23	230	43	991
64	993	84	776	4	550	24	318	44	3694078
65	3687083	85	864	5	639	25	406	45	166
66	172	86	953	6	727	26	494	46	254
67	261	87	3689042	7	816	27	582	47	342
68	350	88	131	8	904	28	671	48	430
69	440	89	220	9	993	29	759	49	517
70	529	90	309	10	3691081	30	847	50	605
71	618	91	398	11	170	31	935	51	693
72	707	92	486	12	258	32	3693023	52	781
73	796	93	575	13	347	33	111	53	868
74	885	94	664	14	435	34	199	54	956
75	975	95	753	15	523	35	287	55	3695044
76	3688064	96	841	16	612	36	375	56	131
77	153	97	930	17	700	37	463	57	219
78	242	98	3690019	18	788	38	551	58	306
79	331	99	107	19	877	39	639	59	394
80	420	4900	196	20	965	40	727	60	[illegible]

Ggg

Figure 7-4: Excerpt from a table by [Frobenius] showing 23 degrees and 31 minutes.

Frobenius leaves it open to the reader as to which computation method might be used. However, given that very extensive and very detailed six and/or seven digit tables for both trigonometric functions and for logarithms were available, there was a preference to use logarithms as the latter were better known.

George Ludwig Frobenius (1566 - 1645)
Link http://de.wikipedia.org/wiki/Georg_Ludwig_Frobenius

Since the 17th Century, proportional calculating instruments, such as the sector and the proportional divider/circle developed by Jost Bürgi [Staudacher], make wide-spread appearances.

Nicholas Rose [Rose] has described further applications of Prosthaphaeresis. In one such, he applies Prosthaphaeresis to music, to explain the theory of vibrations and beats; in another, he explains why it is not possible to receive high-fidelity reproduction using a signal from a medium-wave transmitter.

The prosthaphaeretic method was used for about hundred years with great eagerness, because it represented a considerable aid to computation. Moreover, to judge from the works of Longomontan and Frobenius [Frobenius], several mathematicians continued to use their trusted Prosthaphaeresis even after the publication of the logarithms.

Others however concerned themselves with the logarithms and valued their use very highly: The astronomer and mathematician Marquis Pierre-Simon de Laplace (1749 – 1827) claims that

"The invention of the logarithms shortens calculations which might have lasted for months, to a few days, doubling thereby the life of the (human) computers."

Moreover, the application of Prosthaphaeresis had been able to contribute for some decades and it may have been the trigger for the emergence of logarithms, because John Napier (1550-1617) and Jost Bürgi (1552 - 1632), the first calculators and publishers of logarithms, were trusting users of Prosthaphaeresis. Bürgi used Prosthaphaeresis for his computations concerning his observations of Mars around 1590 [Faustmann]. According to Volker Bialas [Bialas] Johannes Kepler (1571-1630) also used Prosthaphaeresis for his calculations for his "Epitome Astronomiae Copernicanae" (Outline of Copernican Astronomy - 1618/1621).

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