Napier revisited *or* A new look at the computation of his logarithms

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Abstract. After more than 400 years it may seem improbable that something new can be said about the way John Napier (1550-1617) computed his logarithms for the *Mirifici logarithmorum canonis descriptio* of 1614. On the other hand, time has passed over these logarithms and made them as well as the ways of their computation obsolete almost immediately after their publication. Moreover, we had to wait 270 years until a translation of the Latin original of the (posthumous) *Mirifici logarithmorum canonis constructio* of 1619 into English was achieved, and this has remained the only one. This shows clearly that the interest in Napier's work is confined to the realm of history – but even there we find nothing but the facts that are well known for almost two centuries. We propose to follow the path opened by one of us and to have a new and fresh look at the rules and methods, which had to be formulated from scratch by John Napier, as well as their possible applications.

Nach 400 Jahren erscheint es unwahrscheinlich, daß über Napiers Logarithmen und ihre Berechnung noch etwas Neues gesagt werden könnte. Andererseits ist die Zeit rasch über Napiers Logarithmen hinweggeschritten und machte sie und ihre Berechnung nahezu sofort nach Erscheinen obsolet. Man mußte allein schon 270 Jahre warten, bevor die *Constructio* aus dem Lateinischen in Englische übersetzt wurde, und das ist bisher die einzige Übersetzung geblieben. Das zeigt deutlich, daß das Interesse an Napiers Arbeit sich allenfalls auf die Historiker beschränkt – aber auch hier finden wir nichts als die Wiederholung der Fakten, die schon seit fast zwei Jahrhunderten bekannt sind. Wir schlagen statt dessen vor, den Kurs einzuschlagen, der von einem von uns vor einiger Zeit eröffnet wurde, und werfen daher einen neuen und frischen Blick auf die Regeln und Methoden, die Napier aus dem Nichts entwickelte, und auf ihre mögliche Anwendung.

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0. Introduction and Basics

2014 is the quartercentenary of John Napier's (1550-1617) publication of his table of logarithms, together with instructions to use, published in 1614: *Mirifici Logarithmorum Canonis Descriptio* (*Descriptio* for short). Five years later, i.e. two years after Napier's death, the construction manual for his logarithms appeared posthumously in 1619: *Mirifici Logarithmorum Canonis Constructio* (*Constructio* for short). Both works were written in Latin; but whereas an English version of the table and its instruction manual was given as early as 1616 and 1618 by Edward Wright (although for reasons currently unknown to us Wright decided to give the logarithms with one figure less, 6 instead of 7 places), and Ursinus's table of 1624 with 8 places and refined step size between the entries were soon to follow, we had to wait 270 years for a translation of the *Constructio*. The explanation of this neglect – and thus also for a sign of disinterest – is simple: as we all know, Napier himself contributed right after the publication to a new definition of logarithms, which led to the Briggsian or decimal or common logarithms, and these new logarithms almost instantly and completely superseded the original work. Simultaneously, other and new methods were conceived for the computation of these logarithms, and the need to see how Napier achieved *his* computations strongly diminished. Since then, it has been the object of some research by mathematicians and astronomers with historical ambitions, like Delambre or Biot, and some of Napier's (minor) errors were pointed out or elucidated. However, there is still the need to have a closer look. We mention here explicitly Napier's miscalculation of the crucial value of $10000000 \cdot 0.99999^{50}$; he gives 9995001.222927, but this has been corrected to 9995001.224804 already around 1835 by Biot, and at least definitely 1865 by Edward Sang. (The error may look insignificant, but of course, it isn't.) From there it went into Macdonald's translation of the *Constructio* in 1889 and has been repeated ever since. Curiously enough, this corrected value is wrong, too, but this has gone unnoticed for at least more than a century until 1997 (Fischer 1997; Fischer 1998), when the "real true" value of 9995001.224826 was published. (This does not, admittedly, influence the previous error very much, but that is not the point here...) This value for the first time respected Napier's rules of computing with finite precision, which he expressly laid down in the *Constructio*, and this and other observations gave us the impression that a fresh and closer look to that highly praised but at the same time almost totally neglected publication might be in order; and of course we had recourse to the Latin original…

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We will outline now some basics concerning Napier's logarithms. This will be done using modern notions and notations, while at the same time trying to respect what Napier actually says, does or means. The § numbers given are those of the *Constructio* of the first 1619 edition, Edinburgh (Napier 1619, Napier 1620). Of course we will have to skip many details.– Napier first considers the *continuous* motions of two points A and G. Point A is moving with constant velocity along a straight line, starting at B (*arithmetical motion*, § 23). Point G, starting at T, is moving along another straight line towards a fixed point S with a velocity always proportional to its current distance from S; consequently G is permanently slowing down (*geometrically decreasing motion*; §§ 24-25). If A and G start their respective motions at the same time $t = 0$, and if the initial velocities of A and G are the same, then at time $t \geq 0$ the length $\overline{BA} =: y(t)$ is called the logarithm of the remaining length $\overline{GS} =: x(t)$ (§ 26). Let the distance between T and S be called h; then we have $y(0) = 0$ and $x(0) = h$. Napier thus (verbally) defines a functional relationship for his logarithm, which we will call LN: $LN(x(t)) := y(t)$. However, t can be eliminated; and using the synchronising conditions given by Napier (same starting time, same initial velocities) *we* do eventually obtain a *function* $LN(x)$. With hindsight, all this translates into

$$(1) \qquad LN(x) = h \cdot (\ln(h) - \ln(x)) = h \cdot \ln(h/x),$$

where ln has our usual meaning. As it should be (remember $x(0) = h$ *and* $y(0) = 0$), we have

$$(2) \qquad LN(h) = 0$$

(§ 27); and as it takes x an infinite time to become 0 (i.e. $x(\infty) = 0$ *and* $y(\infty) = \infty$), we also have

$$(3) \qquad LN(0) = \infty.$$

If h were 1, then from (1) we would have $LN(x) = \ln(1/x) = -\ln(x)$, and Napier's LN would immediately be recognisable as the logarithm to base e^{-1}. Napier however chose $h = 10^7$, as he wanted a table of LN for the sines of the angles α in the first quadrant, α = 0°0' (1') 90°0', and sines were tabulated in Napier's times as *natural* numbers, e.g. resulting from a radius of $h = 10^7 = 10000000$ (this special h was first chosen by Regiomontanus in the 15th century; remember that the use of a decimal point was yet to be introduced – by Napier). In what follows, sines of this kind will be designated SIN; up to rounding we have $SIN(\alpha) = h \cdot \sin(\alpha)$. Thus Napier's aim was to compute a table of the 5401 values $LN(SIN(\alpha))$, from LN(SIN(0°0')) = ∞ in α-steps of 1' to LN(SIN(90°0')) = 0. Actually the *computation* was done just the other way round, from the finite value LN(SIN(90°0')) = LN(h) = 0 down to LN(SIN(0°1')), i.e. to the last value before LN(SIN(0°0')) = LN(0) = ∞.

In the posthumously published 1619 *Constructio*, LN is considered only in the interval [0,h], although in the 1614 *Descriptio* (Napier 1614, Napier 1620), published first but written later, Napier pointed out that for $x > h$ his LN still has a meaning but will then become negative (we will not deal with this, however, as it doesn't concern the *computation* of Napier's LN table). In the interval [0,h] Napier's LN is monotonically decreasing from $LN(0) = \infty$ to $LN(h) = 0$. From LN's definition in modern terms (1) *we* deduce at once

(4) $$LN(a \cdot b) = LN(a) + LN(b) - h \cdot \ln(h) = LN(a) + LN(b) - LN(1),$$

and because $LN(1) = 161180956.5 \neq 0$, there is no functional equation like $\log(a \cdot b) = \log(a) + \log(b)$. But such a thing, of course, was not what Napier was looking for. Instead, *he* deduced (among several other properties of LN)

(5) $$LN(a) - LN(b) = LN(u) - LN(v), \quad \text{if} \quad a : b = u : v$$

(§§ 35-38), implicitly requiring $0 < a < b \leq h$, $0 < u < v \leq h$ in order to avoid negative numbers and to stay in the original range (0,h], 0 excepted in this case. This was even better suited than a functional equation for Napier and his contemporaries, because they were accustomed to calculating in proportions. (5) will subsequently be used; but it is easy to obtain something very similar to our functional equation: Consider $(h \cdot a) : (h \cdot b) = (h \cdot a/b) : h$, which by (5) and (2) results in

(5a) $$LN(h \cdot a/b) = LN(h \cdot a) - LN(h \cdot b);$$

or consider $(h \cdot ab) : (h \cdot b) = (h \cdot a) : h$, which again by (5) and (2) leads to

(5b) $$LN(h \cdot ab) = LN(h \cdot a) + LN(h \cdot b).$$

Obviously this now requires $0 < a < b \leq 1$, but as (5a) and (5b) nearly reproduce the "classical" (but later) logarithmic functional equation, this suggests or supports the view of h as a mere scaling factor – which indeed it is; see $SIN(\alpha) = h \cdot \sin(\alpha)$ or $LN(h \cdot \sin(\alpha)) = h \cdot \ln(1/\sin(\alpha))$.

1. Napier's problems…

We (probably) know everything about ln and thus about LN; especially we know how to *compute* their values. Napier however *invented* LN and at first only knew that $LN(h) = 0$ (and $LN(0) = \infty$). In order to obtain numerical values for arguments $0 < x < h$, Napier had to think about several problems:

a) Given the situation that it takes G an infinite time to reach S (as shown by $LN(0) = \infty$), he has to find feasible approximations when x is nearing 0; in particular he cannot include or must avoid $x = 0$ in his computations.
b) Consequently, although the *final interval of computation* will be [0,h], Napier will have to choose a smaller interval [g,h], $0 < g < h$, as a *starting interval* (remember that the only finite value he knows for LN is $LN(h) = 0$, so h has to be included).
c) Consequently, again, Napier will have to find means to determine a suitable g and equally suitable procedures that will enable him to move the left end of the *current* interval of computation nearer to 0, possibly step by step.
d) He will also have to define the *accuracy* with which his logarithms will be computed, at least in the starting interval.
e) Moreover, finally, as he has nothing at his disposal to compare his LN with, he has to undertake what we today would call a *forward analysis of the propagation of errors* in order to check if his desired level of accuracy is attained.

One of us has devoted two earlier papers (Fischer 1997, Fischer 1998) to sketching a reconstruction – based, of course, on Napier's *Constructio* – of how Napier might have solved some of these problems. From his writings it is obvious that he had a clear notion of d) and its consequences and requirements, and that for e) he *did* (or even invented) interval arithmetic and eventually embedded a geometrically decreasing *sequence* into his *continuous* starting interval which guaranteed d), but without any hints or details why this is so. Due to the strictly Euclidean style of Napier's publications, many of his important thoughts are easy to overlook or were even suppressed by him. The papers just mentioned were therefore meant, among other things, to show (for the first time, as far as we know) that Napier for his purposes indeed *created* interval arithmetic (because he *had* to, in order to do the forward analysis needed and respecting the finiteness of his numbers); to *explain* why Napier chose 0.9995 as the ratio of two consecutive terms of the embedded geometric sequence; and at the same time to show that and *why* this choice was perfectly appropriate to guarantee the desired level of accuracy. So we will refrain from repeating this here; and we will also refrain from describing Napier's absolutely intriguing interpolation procedure by which he obtains logarithms for numbers not found in the sequence, and why this procedure will also work within the predefined accuracy in the starting interval (Napier's interpolation scheme in fact corresponds to what we today would call linear Taylor approximation, but in the special case of LN it was introduced into mathematics by Napier three generations *before* Taylor and was proved, of course, verbally/kinematically/geometrically).

Speaking of geometrically decreasing sequences, we observe the following facts: Napier's sequences – for in fact there are a few of them – all start at h and have quotients obeying $0 < p, q, \ldots. < 1$. The members of any pair of consecutive terms thus have the same ratio: $(h \cdot q^n) : (h \cdot q^{n-1}) = (h \cdot q^{n-1}) : (h \cdot q^{n-2}) = \ldots. = (h \cdot q^1) : (h \cdot q^0) = q$. From (5) we have

(6a) $LN(h \cdot q^n) - LN(h \cdot q^{n-1}) = \ldots = LN(h \cdot q^2) - LN(h \cdot q) = LN(h \cdot q) - LN(h)$

(§ 36), which, by using (2) and going backward through these equations, leads to

(6b) $LN(h \cdot q^n) = n \cdot LN(h \cdot q)$.

This has an analogue also for geometrically decreasing double sequences. Again by (5), and starting with $(h \cdot q^n p^m) : (h \cdot q^n) = (h \cdot p^m) : h$, we first have

(6c) $\quad LN(h \cdot q^n p^m) - LN(h \cdot q^n) = LN(h \cdot p^m)$

(we also could have used (5b) directly); then, by applying (6b), we obtain (with Napier)

(6d) $\quad LN(h \cdot q^n p^m) = n \cdot LN(h \cdot q) + m \cdot LN(h \cdot p)$.

This will also be used later; but now let us first have a look at the two solutions presented by Napier for problems a) to c).

2. …and two of his solutions

Method I. Consider two numbers related to each other by $a : b = 1 : 2 = \frac{1}{2}h : h$, in particular $\frac{1}{4}h \le a < \frac{1}{2}h$ and thus $\frac{1}{2}h \le b < h$. Applying (5) and (2) results in

(7a) $\quad LN(a) - LN(b) = LN(a) - LN(2a) = LN(\frac{1}{2}h) - LN(h) = LN(\frac{1}{2}h)$, or

(7b) $\quad LN(a) = LN(2a) + LN(\frac{1}{2}h)$

(§ 51). Thus, if $LN(\frac{1}{2}h)$ and $LN(2a) = LN(b)$ were known, then Napier would also know $LN(a)$, in particular for $\frac{1}{4}h \le a < \frac{1}{2}h$. The special case $a = \frac{1}{4}h$ leads to

(7c) $\quad LN(\frac{1}{4}h) = LN(\frac{1}{2}h) + LN(\frac{1}{2}h) = 2 \cdot LN(\frac{1}{2}h)$.

Consider now two numbers related to each other by $a : b = 1 : 4 = \frac{1}{4}h : h$, in particular with $\frac{1}{8}h \le a < \frac{1}{4}h$ and thus again $\frac{1}{2}h \le b < h$; then the same reasoning as above, but additionally using (7c), results in

(8a) $\quad LN(a) - LN(b) = LN(a) - LN(4a) = LN(\frac{1}{4}h) - LN(h) = LN(\frac{1}{4}h)$, or

(8b) $\quad LN(a) = LN(4a) + LN(\frac{1}{4}h) = LN(4a) + 2 \cdot LN(\frac{1}{2}h)$.

If $LN(\frac{1}{2}h)$ and $LN(4a) = LN(b)$ were known, then Napier would now also know $LN(a)$, in particular for $\frac{1}{8}h \le a < \frac{1}{4}h$. The special case $a = \frac{1}{8}h$ leads to

(8c) $\quad LN(\frac{1}{8}h) = LN(\frac{1}{2}h) + 2 \cdot LN(\frac{1}{2}h) = 3 \cdot LN(\frac{1}{2}h)$.

Continuing this way, we obtain in general

(9) $\quad LN(a) = LN(2^k a) + k \cdot LN(\frac{1}{2}h)$,

in particular applied to an a with $h/2^{k+1} \le a < h/2^k$, thus lifting $2^k a = b$ into $\frac{1}{2}h \le b < h$. This suggests the following solution of problems 1.a) to 1.c):

a) Define $[\frac{1}{2}h, h]$ as the *starting interval*.
b) Compute the logarithms of all the numbers you need (using the interpolation procedure mentioned above), if these numbers are falling into the starting interval. This in particular includes $LN(\frac{1}{2}h)$.
c) For numbers $0 < a < \frac{1}{2}h$ there is usually only one multiplier 2^k which lifts $2^k a = b$ into the starting interval $[\frac{1}{2}h, h]$; then apply (9) to obtain $LN(a)$.

To be precise, c) is not exactly what Napier does: As a tribute to the decimal system, he instead uses the 28 multipliers m = 2, 4, 8, 10, 20, 40, 80, 100, … 8000000, 10000000 (§ 53) and computes the constants c_m to be added to $LN(m \cdot a)$ when using m as multiplier (§§ 52-53), just like in (9). It is easy to see that this can be done in a similar way by also using $LN(\frac{1}{10}h)$. As $\frac{1}{10}h$ is not in the starting interval, however, Napier has to do one (easy) extra computation. The c_m of course are linear combinations of $LN(\frac{1}{2}h)$ and $LN(\frac{1}{10}h)$ with natural coefficients. This obviously does not differ much from c), although in many cases there are now *two* choices for the multiplier.

If executed systematically, this procedure allows to extend the interval of computation to the left, step by step, starting with $[\frac{1}{2}h, h]$, then (k = 1) joining the interval $[\frac{1}{4}h, \frac{1}{2}h)$, then (k = 2) the interval $[\frac{1}{8}h, \frac{1}{4}h)$, and so on. If, however, executed with the knowledge of *all* the multipliers beforehand, Napier can at once complete his table: he just has to take the correct multiplier and then to proceed as usual. There is no restriction on the values of a except $0 < a < \frac{1}{2}h$. Specifically, this procedure does *not* require that a = SIN(α) for some α appearing as an argument in the SIN table; instead it will work for any $0 < a < \frac{1}{2}h$ and thus is more general than perhaps needed for the construction of a LN(SIN(..)) table.

Method II. It is obvious from his writings that Napier started with Method I. However, it also seems that at some time – probably near the end of his computations – he was in doubt whether this procedure would always lead to reliable results. Maybe originally he had in mind a general procedure for logarithms LN(x), regardless of whether x is the SIN of some angle α or not. But for his alternative method he now made use of a special property of sines/SINes (§ 55), which in modern notation is given by

$$\text{(10a)} \qquad \sin(\alpha) = 2\sin(\frac{\alpha}{2})\sin(90° - \frac{\alpha}{2}).$$

Due to the scaling factor $h = 10^7$ connecting sines and SINes, and also due to writing equations in the form of proportions, Napier formulated and geometrically proved this property as

$$\text{(10b)} \qquad \frac{\mathrm{SIN}(90°)}{2} : \mathrm{SIN}(\frac{\alpha}{2}) = \mathrm{SIN}(90° - \frac{\alpha}{2}) : \mathrm{SIN}(\alpha).$$

This immediately translates, once again by (5), into Napier's logarithms LN:

$$\text{(11a)} \qquad \mathrm{LN}(\frac{\mathrm{SIN}(90°)}{2}) - \mathrm{LN}(\mathrm{SIN}(\frac{\alpha}{2})) = \mathrm{LN}(\mathrm{SIN}(90° - \frac{\alpha}{2})) - \mathrm{LN}(\mathrm{SIN}(\alpha));$$

by using SIN(90°) = h and SIN(90°)/2 = $\frac{1}{2}h$, this results in

$$\text{(11b)} \qquad \mathrm{LN}(\mathrm{SIN}(\frac{\alpha}{2})) = \mathrm{LN}(\mathrm{SIN}(\alpha)) - \mathrm{LN}(\mathrm{SIN}(90° - \frac{\alpha}{2})) + \mathrm{LN}(\tfrac{1}{2}h)$$

(§ 57). The special case α = 90° (§ 56) leads to

$$\text{(11c)} \qquad \mathrm{LN}(\mathrm{SIN}(45°)) = \mathrm{LN}(h) - \mathrm{LN}(\mathrm{SIN}(45°)) + \mathrm{LN}(\tfrac{1}{2}h), \text{ or}$$
$$\text{(11d)} \qquad \mathrm{LN}(\tfrac{1}{2}h) = 2 \cdot \mathrm{LN}(\mathrm{SIN}(45°)).$$

Thus, by (11d), if LN(SIN(45°)) is known, then also $\mathrm{LN}(\frac{1}{2}h)$ is known, as it is the same. And in general, if the logarithms LN(SIN(45°)) ... LN(SIN(90°)) are known, Napier is then able to compute, by (11b), LN(SIN(α/2)) for all these α, 45° ≤ α ≤ 90°, because he also has 45° ≤ 90°–α/2 < 90°. So in the first step of Method II he could have obtained his LNs for SINes down to LN(SIN(22°30')). Continuing this way, he would extend this down to LN(SIN(11°15')), then down to LN(SIN(5°38')) and so on (§ 58). (This is mirrored by what was called above a "systematic execution" of Method I.) At the same time the starting interval is substantially reduced from $[\frac{1}{2}h, h]$ to $[\frac{\sqrt{2}}{2}h, h]$.

Extending (11b) by iterating k times and setting $\alpha/2^k = \beta$, *we* obtain (Napier doesn't do this)

$$\text{(11e)} \quad \mathrm{LN}(\mathrm{SIN}(\beta)) = \mathrm{LN}(\mathrm{SIN}(2^k\beta)) - \sum_{i=0}^{k-1} \mathrm{LN}\,(\mathrm{SIN}(90°-2^i\beta)) + k \cdot \mathrm{LN}(\tfrac{1}{2}h).$$

E.g., for β = 5° and k = 3 (which suffices to reach the original starting interval; k = 4 would get us into the reduced starting interval) we will only have to do the following easy calculation: LN(SIN(5°)) = LN(SIN(40°)) – LN(SIN(85°)) – LN(SIN(80°)) – LN(SIN(70°)) + 3 · LN($\tfrac{1}{2}$h), in numbers: LN(SIN(5°)) = 4419408 – 38126 – 153088 – 622025 + 3 · 6931469 = 24400576, where all the LNs needed are taken from Napier's table. There, however, we find LN(SIN(5°)) = 2440057<u>8</u>, but this would have been the result if we would have taken LN values with one additional figure after the decimal point (and these numbers were available to Napier and were used by him, but of course remained unpublished).

Method II vs. Method I. While Method I seems more general (and under usual circumstances indeed is – but Napier's circumstances obviously were *not* usual), Method II has three distinctive features worth mentioning:

a) As already pointed out, the starting interval is considerably smaller; in terms of SIN values it is [7071068,10000000] instead of [5000000,10000000]. An embedded geometrically decreasing sequence with ratio 0.9995, starting at the right end h = 10^7, has reached the left end 7071068 of the starting interval after only 693 steps, compared to 1386 steps which are necessary to reach 5000000.
b) By concentrating on LN(SIN(..)), the structure of the SIN table used is implicitly transferred to logarithms, especially the given subdivision of its arguments into degrees and minutes. This is exactly what Napier was aiming for, but brings a certain loss of generality with it. (To see this, simply suppose LN(x) should be needed when x is *not* exactly a SIN value, then one would have to do several rather tedious interpolations.)
c) But the most striking and intriguing feature of Method II is clearly shown by (11b): In order to compute LN(SIN(α/2)), the *value* of SIN(α/2) *doesn't even have to be known*!

Napier's preference. Napier suspected (§ 60) that the SIN tables at his disposal were not free from errors (and of course he was right). Because Method II theoretically enabled him *not to use* any of the SIN values below 45° (but he never mentions this in the *Constructio*; in the *Descriptio* we find it hidden in Book I, Chapter V, at the end of Problem 3: *absque sinibus*, but one has to know what one should be looking for), he had a clear preference for Method II, although he obviously started the whole thing – and therefore most of his calculations – with Method I in mind. It seems that only after having observed some discrepancies that Napier thought about another solution.

His preference for Method II becomes clear when we consider his proposal – made at the very end of the *Constructio*, in § 60 – for the construction of a more precise table of LN, taking h = 10^8 instead of h = 10^7. Here we read that the ratio of the embedded geometrically decreasing sequence should be taken as 0.9999 (which, by the way, is in perfect accordance with the reconstruction mentioned above, explains this choice, and therefore also guarantees Napier the accuracy requested), that the starting interval should be taken as [70710678,100000000] and that accordingly Method II should be used to extend the table.

3. More problems…for us

As we have only a few hints – probably written at different times, too – as to how Napier's calculations were carried out, let's now look at what else we have: Napier's 1614 table for LN, as published in the *Descriptio*. As usual with mathematical tables, we find errors of several kinds.

We already mentioned above that Napier first decided in favour of an embedded geometrically decreasing sequence with $q = 0.9995$, which would have brought him in 1386 steps from h to $\frac{1}{2}h$. But in order to facilitate his calculations he observed that $0.9995^{20} \approx 0.99 =: p$ and replaced the original sequence by a double sequence $x_{n,m} := h \cdot q^n p^m$ ($0 \le n \le 20$, $0 \le m \le 68$). This sequence has 1449 members, but due to $q^{20} \approx p$ we have $x_{20,m} \approx x_{0,m+1}$, so 1380 of its members will replace the 1386 terms of the original sequence determined by q alone. Napier found (see (6d)) that

$$(12) \qquad LN(x_{n,m}) = n \cdot LN(h \cdot q) + m \cdot LN(h \cdot p) = n \cdot LN(9995000) + m \cdot LN(9900000)$$

(§§ 46-47). First of all, due to a long identified but still *un*explained computational error, Napier computed the crucial value at $x_{1,0} = 9995000$ as LN(9995000) = 5001.2**485387** (boldface: wrong figures), whereas – in accordance with his own computational rules – he should have obtained 5001.2504386 (the true value being 5001.2504168|224...). Regrettably, this error also affects the computation of LN(x) at $x_{0,1}$, i.e. of LN(9900000). This means that in the end Napier has

$$LN(x_{n,m}) = n \cdot 5001.2\mathbf{485387} + m \cdot 100503.3\mathbf{210291}$$

(boldface: wrong figures), but if the error had not been committed, he would have obtained

$$LN(x_{n,m}) = n \cdot 5001.2504386 + m \cdot 100503.3590591,$$

which is very near to our modern values:

$$LN(x_{n,m}) = n \cdot 5001.2504168 + m \cdot 100503.3585350.$$

Therefore, in general his LN will be systematically smaller than expected. The error in LN(9995000) accumulates (multiplication by 1380, the number of steps to get from h to $\frac{1}{2}h$ by using the double sequence) to an absolute error of $-2.8 \approx -3$ at $x = 5000000 = \frac{1}{2}h$; consequently, in Napier's table we find e.g. $LN(\frac{1}{2}h)$ = LN(SIN(30°)) = LN(5000000) = 6931469 instead of 6931472. But we know from (9) and (11b) or (11e) that $LN(\frac{1}{2}h)$ appears in *both* Methods I and II of extending a starting interval to the final interval of computation, so the error in $LN(\frac{1}{2}h)$ will propagate further throughout Napier's LN table, regardless of the method used. This means that Napier's LN values should be *proportional* to the modern ones with a scaling factor of, say, 5001.2**485387**/5001.2504168 = 0.999999624… This, however, is not yet quite true: due to Napier's use of the double sequence, there is no such single "unifying scale factor" – the factor depends in its ninth decimal figure very slightly on x, because in the end the error in LN(9900000) will prevail. For practical purposes and comparisons a value of 0.999999627 ≈ 100503.3**210291**/100503.3585350 will be better.

Second, Napier compared Methods I and II in § 60, but the only example he gives (undoubtedly he had quite a few others) is for x = 378064. Now this is SIN(2°10') as one can find it in most of the contemporary SIN tables, but Napier doesn't even tell us that. He simply states – in the *very last* paragraph of the *Constructio* – that the LN value obtained by Method I, which he explicitly had computed earlier as LN(378064) = 32752756 (in the example of § 54), differs from the one obtained by Method II, given by him as 32752741, by intolerable 15 units. However, everything concerning the value 32752741 is left to the reader; Napier neither bothers to explain why *this* value – and not the earlier 32752756 – should be the correct one (which indeed it nearly is), nor does he give any further details concerning its computation. So it is left to the reader whether he finds out that 378064 belongs to 2°10', and that LN(SIN(34°40')), which Napier had explicitly computed earlier (in the example of § 57), leads to LN(SIN(2°10')) by 4 times consecutively halving the angle; compare (11e). But only this observation makes the different results comparable!

It was made explicit in the papers mentioned above (Fischer 1997 and Fischer 1998) why such differences are inevitable, and that they *must* appear when *rounded* SIN values have to be used in Method I. For even if all SIN values were correctly rounded, their rounding error of at most ±0.5 is also multiplied by 2^k (or the corresponding factor m in Napier's table of multipliers, which in the example given by him obviously is 20), and thus in the general case does not lead to the correct multiple of $2^k h \cdot \sin(\alpha)$ which would be needed to obtain results within the accuracy required.

Napier has chosen an absolutely perfect example to illustrate his case: First, the value x = 378064 indeed results from *in*correct rounding in the SIN table(s) used; with two more figures we have h · sin(2°10') = 378064.55, so the SIN tables should have given 378065. Second, because of the multiplier m = 20 the error has grown from –0.55 to –0.55 · 20 = –11 after lifting 378064 into the starting interval, obtaining b = 7561280. Third, b's error Δb = –11 produces an LN error of ≈ –h · Δb/b ≈ +14.53 ≈ +15 units, exactly as found by Napier.

Method II however, as pointed out above, *would not even use* the SIN values below 45°, and so definitely avoids this problem. But at which point during his calculations did Napier detect that Method I was not fully reliable? (No answer to this question will be given in what follows, or can probably ever be given.) Did he switch to Method II already from 45° downward, or only from 30° downward (because down to this angle he had already extended this original starting interval), or from another angle? Here we will later provide partial answers.

However, third, at least in the starting interval Napier has to work with the values as they are given in the SIN table. Theoretically there were three good candidates with $h = 10^7$ at Napier's disposal when he presumably began to work on his logarithms: Reinhold 1554, Finck 1583 and Lansbergen 1591 (see e.g. Glowatzki and Göttsche 1990). With regard to $h = 10^7$ it has become customary – and seems to be fair, too – to consider any SIN errors of absolute value > 2 as printing or printer's errors (although this may be debatable in a very few instances), whereas errors of ±1 and even of ±2 should be considered either rounding errors (if ±1) or minor approximation errors (if ±2), or both, but should be tolerated or interpreted as still being exact. This is in accordance with the fact that contemporary *ideal* tables should of course have given only correctly computed and correctly rounded (natural) numbers as their result, but also with taking into account that in the 16th and 17th centuries this was formulated as meaning something like "the (absolute) error should be smaller than (or at most) one unit of the last figure given in the table", so an error of ±1 (instead of ±0.5) was acceptable and not considered an error at all. As Glowatzki & Göttsche have shown in 1990, all of the three candidates mentioned above are based on Regiomontanus's SIN table, posthumously published in 1541 by Schöner; however, the number of what are considered printing errors in the sense described above has been considerably reduced from 93 (Regiomontanus 1541) to 15 (Reinhold 1554) or 18 (Finck 1583 *and* Lansbergen 1591, the latter two being identical according to Glowatzki & Göttsche, after all they tested). 11 among the 15 or respectively 18 printing errors left uncorrected are identical; consequently there are 4 such errors appearing only in Reinhold's table, but not in Finck's or Lansbergen's table, and 7 appearing only in Finck's or Lansbergen's table, but not in Reinhold's. (The comparisons made by Glowatzki & Göttsche, however, do *neither* detect newly introduced printing errors, *nor* do they detect other differences in the SIN values, as this was not in their line of investigation. Naturally enough there *are* new printing errors… but this would be another story.)

And this, fourth, raises the next question: Which one of these three tables was actually *used* by Napier for his computations (and how did he deal with the errors in these tables)? Unlike Finck and Lansbergen, Reinhold gave differences for his SINes, so in case an error was suspected it was a little bit easier to detect and to correct. But according to the findings of Glowatzki & Göttsche, Napier's 1614 table – in which not only LN(SIN(α)), but also SIN(α) is given – shows in its SIN part exactly the same 18 printing errors as Finck and Lansbergen, from which observation Glowatzki & Göttsche concluded that Napier must have *used* either Finck's or Lansbergen's SIN table, and this view has been accepted since (e.g. Lüneburg 2008, Roegel 2010). But can we really be sure about that? Of course the answer is "no" (see VIII.).

4. What *would* we expect? What *can* we expect?

Napier is clear about the rules that have to be obeyed during computation. He tells us precisely how many figures are to be used, whether the result of a multiplication or a division has to be rounded or truncated, and in the latter case where, when and how truncation is done. Thus there should be a good chance to be able to (almost exactly) reproduce Napier's table of LN(SIN(α)), simply by following his recipes. Alas, it is not as simple as that, even if one accepts deviations of ± 1 or at most ± 2. One thing, however, should be made clear: one must not re-do Napier's calculations by today's standards and compare the two; especially not when higher precision is used and when at the same time Napier's rules are disregarded. However, exactly this seems to be tempting… from the last third of the 19th century (Sang, Macdonald: see Napier 1889/1966, *passim*) to our days (Roegel 2010).

There is a simple reason why this does not make sense. As already pointed out, Napier made an impeccable forward analysis of the propagation of errors, using interval arithmetic. Now on the one hand forward analysis tends to overestimate the errors, but on the other hand Napier's initial value for LN(9999999), which he took to be 1.00000005, has not only precision 1/h as required by his forward analysis, but only $2/h^2$ (as an easy analysis shows; see Macdonald in Napier 1889/1966). It takes about 6.9 million ($\approx$ 0.7 h) steps with factor 0.9999999 to get from h to $\frac{1}{2}$h ; Napier wanted LN($\frac{1}{2}$h) to have an error of at most ± 1, but in reality – given the precision (unknown to him, of course) of $2/h^2$ in LN(9999999) – he attained a much, much better precision of $0.7 \cdot 2/h \approx 1.5/h$ (not regarding some other technicalities here). This is also what we see when computing a scaling factor between Napierian logarithms and h $\cdot$ ln(h/x) similar to the one given above, but this time not for Napier's logarithms with the error introduced by him, but with the values he should have obtained without. This factor turns out to be 1.0000000052 = : s, and will produce an error of +1 (because in this case Napier's logarithms would have been bigger than the true LN would be) only at LN(x) = 192218768, because approximately (at first not taking into account the effects of rounding) we must have s $\cdot$ LN(x) = LN(x) + 1 or LN(x) = $\frac{1}{s-1}$. But the largest value in Napier's LN table is 81425681 = LN(SIN(0°1')), and thus is *less than half this critical value*. This shows that all of Napier's LNs would have had an error of less than ±0.5 even if they are "near to infinity" (i.e. near to LN(SIN(0°0')) = ∞) – if only Napier wouldn't have committed his central error.

By the way, we can use the same reasoning to see when Napier's error-loaded logarithms will theoretically first produce a deviation of –1 (because now they are smaller than the correct LN): The factor s = 0.999999627 will do this when s · LN(x) = LN(x) – 1, or LN(x) = $\frac{1}{1-s}$, which results in LN(x) = 2662931. But this already is the case for x ≈ SIN(50°0'), and due to rounding this will happen several degrees upward, i.e. when s · LN(x) ≈ LN(x) – 0.5, or LN(x) ≈ 1/2(1 – s), or LN(x) ≈ 1331466, or x ≈ SIN(61°5'). Indeed we observe that from 60°51' downward we have a deviation of –1 (after rounding), and this grows in size until it has reached –30 at the angle nearest 0 (where LN is nearest to infinity), which of course again is 0°1'.– This leads us to a closer inspection of the numeric behind Napier's computation.

I.

How does the SIN table influence Napier's LNs? First of all, let us consider only the original starting interval [$\frac{1}{2}$h,h] and let us postpone the question whether Napier used Reinhold's, Finck's or Lansbergen's tables. For the time being we will assume, contrary to Glowatzki & Göttsche 1990, that he used Reinhold's, because on the one hand we have some evidence that Napier initially *did* use it (see below VIII.), and on the other hand for the simple reason that Reinhold's table has already been digitised, whereas Finck's and Lansbergen's haven't been yet. As mentioned above, in Reinhold's table 15 printing errors (i.e. deviations of absolute value > 2 in his SIN values compared to the correctly rounded ones) have been identified by Glowatzki & Göttsche as having remained of the 93 errors of Regiomontanus's 1541 table. Reinhold and/or his printer also *introduced* 15 *further* values with deviations of size > 2. We decided to assume that Napier would have identified any such errors and consequently we eliminated these errors by the following procedure: If a single figure of rank > 0 can be identified as causing the (main) error, it is replaced by the correct one. If necessary, this has to be iterated. If there remains the correct value or an error of absolute value ≤ 2, nothing further is done. In just two cases deviations of +3 and +4 remained (i.e. the last figure, rank 0, was slightly, but > 2, wrong); in these two cases Regiomontanus's value was taken instead. After these corrections, we have the following result: Out of 5401 SIN values, 3560 have no error when compared to correctly rounded modern values; 1825 have errors of ±1 (1211 have +1, 614 have –1) and 16 have errors of ±2 (14 have +2, 2 have –2). There is a slight but clear bias for positive errors. The same count for the angles between 30° and 90°, corresponding to the starting interval [$\frac{1}{2}$h,h], has the following result: Out of 3601 SIN values, 2364 are free from errors, 1226 have errors of ±1 (815 have +1, 411 have –1), and 11 have errors of ±2 (10 have +2, 1 has –2); i.e. the relations seem to be stable, but again show the mentioned bias.

Using (ideally) LN'(x) = –h/x *we* can find that

(13) $\quad$ LN(x+Δx) ≈ LN(x) – Δx · h/x = LN(x) – h · Δx/x,

as LN''(x) = h/x² etc. can obviously be neglected for $\frac{1}{2}$h ≤ x ≤ h. In passing we observe that in this interval we have 1 ≤ h/x ≤ 2; –h/x is the factor by which Δx is enlarged. If x = SIN(α) has been correctly rounded, then 0 ≤ |Δx| ≤ 0.5 and consequently 0 ≤ |Δx · h/x| ≤ 1. In most instances, the upper limit 1 is rather sceptical and will not be attained, but nonetheless tells us that even in these cases the SIN values available to Napier, though rounded, will generate only *tolerable* errors in LN (but it is equally obvious that a correctly rounded SIN value does not necessarily or even automatically produce a correctly rounded LN value). If, however, SIN(α) deviates from its correctly rounded value by ±1 or even by ±2 (which is the case – these two errors combined – for more than one third of the SIN values in the starting interval), the situation changes. For an error of ±1 after rounding means that the true error is *at least* ±0.5 and at most ±1.5, and consequently the error introduced into LN before rounding is at least also ±0.5, but can attain values, in particular for x near to $\frac{1}{2}$h, up to ±3. It is obvious that each of the (few) errors of ±2 makes the situation even worse, introducing at least an error of ±1.5, but up to ±5 for x near to $\frac{1}{2}$h. And because two thirds of the deviations in Reinhold's SIN have a positive sign, we expect that two thirds or more of the deviations in LN (compared to modern values) should have a negative sign, the more so as Napier's error in LN(9995000) also leads to his LN being systematically smaller than the true LN.

After excluding two obvious (printer's?) errors in LN(SIN(45°11')) = 3433839 (Napier's table has 34338**2**9) and in LN(SIN(45°12')) = 3430949 (Napier's table gives 343094**0**), and a rather curious sequence of four consecutive errors of the same magnitude, but with different sign (LN(SIN(33°31')) through LN(SIN(33°34') should have been 5938819, 5934428, 5930040 and 5925655, but in the table we read 59388**2**9, 59344**3**8, 59300**5**0 and 59256**6**5), there are no deviations less than –6 or greater than 3 from the correct modern LN values in Napier's table (as far as the *starting interval* is concerned, of course) and we have 368 deviations > 0, 2201 deviations < 0, and 1032 correct values, which means that a little bit less than 29% of the LN values in the starting interval as given by Napier are correct, about 61% have negative deviations, and a little bit more than 10% positive ones. If we compare this to the percentage of SIN values with corresponding deviations, we there find almost 66% correct values, almost 23% positive deviations (bound to generate negative deviations in LN) and a little bit more than 11% negative deviations. In other terms: we find a substantial change in percentage which may be explained by the tendency (caused by Napier's error at x = 9995000) to underestimate LN's true value. 11% negative deviations in SIN have led to 10% positive deviations in LN, which still seems OK, but 66% correctly rounded SIN values have been reduced to only 29% correctly given LNs, and at the same time the 23% positive deviations of SIN have almost tripled compared to 61% negative deviations in LN. This increase of about 38% is of the same magnitude as the loss of 37% in correctly rounded values.

This can be made even more clear by comparing Napier's LN with the h · ln(h/x) values scaled by the factor 0.999999627 (see above), mimicking Napier's errors at LN(9995000) and LN(9900000). By using these scaled values for comparison we account for the "underestimating tendency" introduced by Napier's errors in LN(9995000) and LN(9900000); what remains should then be (more or less) the errors introduced by Napier having to use *rounded* SIN values from whoever's table. For the starting interval, we now find 1521 (previously 1032) of 3601 LN values to be correct (in this sense), 795 (previously 368) at least one unit too high, 1285 (previously 2201) at least one unit too low; in percentages we now have 22% of the LN values too big, a little bit more than 42% correct and the remaining (a little bit less than) 36% too small. The increase in the percentage of the respective deviations from the correct values, compared to the deviations in the underlying SIN table (11% too small SIN values vs. 22% too big (scaled) h · ln(h/x) values, i.e. an increase of 11%, respectively 23% too big SIN values vs. 36% too small (scaled) h · ln(h/x) values, i.e. an increase of 13%) is distributed almost evenly (11% or 13%, respectively) among both positive and negative deviations – as one would expect by the randomness of the figures after the decimal point in the exact SIN values combined with the rounding rules; this is what we now can see.

But this still isn't a fair comparison, because it doesn't take into account that Napier had no correctly computed SINes with unlimited precision at his disposal, but instead had to rely on somebody else's rounded SINes. If the comparison should be really fair, we would have to compare Napier's values with the true LNs, but *calculated for the* rounded SINes *given in the table used*, and after that scaled down, accounting for Napier's error in LN(9995000). If we do this, we find even more correctly computed LNs, namely 1669 (instead of 1521), 985 (instead of 1285) which are too small, and 947 (instead of 795) which are too big. Translated into percentages we now find more than 46% of Napier's values to be correct (within the set-up described), more than 27% too small and more than 26% too big. But this is exactly what we should expect: because comparison is now made between data which really can be compared, we find that the number of positive and of negative deviations – introduced by rounding – is nearly the same, and that the percentage of Napier's LNs which have no error is now with 46% the highest we find in any of our comparisons.

Termed and counted differently: in the starting interval of 3600 values we have 88.6% of Napier's LN values in perfect accordance (1669) or within ±1 (791 having –1, 730 having +1) of the fair comparison; including a tolerance of ±2 (158 having –2, 169 having +2) this even increases to 97.7%.

II.

A more than merely tentative answer can be given to the question "Did Napier use Method II from 45° or only from 30° downward, or only from an even smaller angle downward?" From (11b), we have

$$\text{(14)} \qquad \mathrm{LN}(\mathrm{SIN}(\frac{\alpha}{2})) + \mathrm{LN}(\mathrm{SIN}(90° - \frac{\alpha}{2})) - \mathrm{LN}(\mathrm{SIN}(\alpha)) - \mathrm{LN}(\tfrac{1}{2}\mathrm{h}) = 0.$$

So it is easy to check – once Napier's 1614 LN table has been digitised – in which cases, i.e. for which values of α, $0° < \alpha \leq 45°$, this equation holds. (Well, it should always hold if the correct values of SIN and LN were available with unlimited or at least with sufficient precision, but neither of these is the case with Napier.) But again a) we only have the printed values, and b) they are natural numbers, i.e. in the best case they have been correctly rounded (because we can fairly be sure that in this computation Napier would have used the LNs available to him with at least one place after the decimal point; see above the example of LN(SIN(5°))). Considering this, one is again led to the conclusion that errors of absolute value ≤ 2 are acceptable in any case, and be it only because each of the 4 terms in (14) would bring a maximal error of ±0.5 with it, assuming that everything else has been correctly computed. This leads to the following summary result:

Deviation	0	±1	±2	> 2	Σ
$0° < \alpha < 30°$	908	500	225	166	1799

$30° \le \alpha < 45°$	238	402	155	105	900
$0° < \alpha < 45°$	1146	902	380	271	2699

While the percentage of deviations > 2 does not show a significant difference and remains in the order of magnitude of 10%, we have more than 50% of the LNs below 30° exactly reproducing (14), but only about 26% of the LNs between 30° and 45°. Therefore, there is strong evidence that Napier used Method II at the earliest from 30° downward, although he could have been using it already from 45° downward.

Another hint is given by Napier himself, although very indirectly: To illustrate Method II, he computes LN(SIN(34°40')) by using LN(SIN(69°20')) = LN(SIN(2 · 34°40') and LN(SIN(55°20')) = LN(SIN(90°–34°40')); his (correct) result is 5642242. But SIN(34°40') is also in the starting interval, so he (and we) can compute LN(SIN(34°40')) using the interpolation procedure already mentioned. Doing this, we find 5642241 (which is correct, too!), and *this* is the value we see in Napier's table, and *not* the 5642242 of Method II. So it is clear that in this special case of an angle in the interval $30° \le \alpha \le 45°$, where ideally he had two choices, Napier stuck to his original interpolation procedure; and it is also obvious now why test (14) is violated in this special case with the result –1.

III.

What else can we find out about the values of Napier's LN in the interval $0° < \alpha < 30°$? A partial answer has been given above: it is most likely that many of these were computed according to Method II, because more than 50% exactly fulfil the identity (14), and more than 90% do it within a tolerance of ±2. But let us now have a closer look at identity (14) itself – or, to be precise, equation (11b), from which (14) was derived by simply putting all its terms on one side –, but this time combined with the properties of the LN table which have been lifted there from the SIN table, namely the subdivision of the angles in the first quadrant into degrees and minutes. If we replace $\alpha/2$ by β, equation (11b) becomes

(15) $$\mathrm{LN}(\mathrm{SIN}(\beta)) = \mathrm{LN}(\mathrm{SIN}(2\beta)) - \mathrm{LN}(\mathrm{SIN}(90°-\beta)) + \mathrm{LN}(\tfrac{1}{2}h).$$

This shows that when computing LN(SIN(β)) from LNs obtained earlier, those LNs belonging to SINes of angles with an *odd* number of minutes and of degrees less than 30° are *never used again* in further computations. This means that – theoretically – not only SINes of angles < 30° are not needed, but also that LNs of SINes for angles < 30° having an odd number of minutes don't play a role, once they have been computed – which also means that possible errors in their computation are not forwarded.

IV.

There is a most peculiar range of angles worth mentioning: the interval $0° < \alpha \le 10°$. In this interval of 600 values, we find *only* 45 angles (or a mere 7.5%) *violating* the exact fulfilment of (14):

0°40', 1°0', 1°30', 1°43', 1°47', 1°48', 1°54', 1°57', 1°58', 2°7', 2°58', 2°59', 3°10', 3°26', 3°34', 3°35', 3°36', 3°37', 3°53', 3°54', 3°56', 4°0', 4°10', 4°14', 4°30', 4°40', 5°0', 5°10', 5°56', 5°57', 5°58', 6°30', 6°40', 7°20', 7°30', 7°40', 8°10', 8°19', 8°20', 8°29', 8°30', 9°20', 9°30', 9°40' and 10°0'.

This already is most astonishing, because it means that 92.5% of the LNs given in this range by Napier fulfil (14) even if they are tested only with the *natural* numbers given by Napier. As we have to consider that Napier most probably used one extra place (the one after the decimal point), so rounding effects would appear, then this is far more than could have been expected.

The second peculiarity of this range is its contrast to what follows. Beginning with 10°1', we find 46 violations of (14) among the 60 values until 11°0', i.e. almost exactly the same number of violations in an interval comprising just $1/10^{th}$ the size of $0° < \alpha \leq 10°$! And it goes on like this; although there are still several exact fulfilments of (14) to come, this sudden and also most extreme change must come as a surprise.

Moreover, finally, there is a third peculiarity of the interval $0° < \alpha \leq 10°$. If we have a closer look at the 45 violations of (14), we find that an astonishing number of 22, i.e. almost half of the 45 violations, occur at angles having a multiple of 10 in their minutes:

0°40', 1°0', 1°30, 3°10', 4°0', 4°10', 4°30', 4°40', 5°0', 5°10', 6°30', 6°40', 7°20', 7°30', 7°40', 8°10', 8°20', 8°30', 9°20', 9°30', 9°40' and 10°0'.

This again is a significantly higher percentage than we would expect. And it seems worth mentioning that these 22 violations consistently are +1, whereas we also find other values for the remaining 23 angles. Therefore, this too looks as if there is a system behind these findings. Concerning another observation which will follow just below, we note that out of the 60 multiples of 10' in the interval $0° < \alpha \leq 10°$, 22 (or almost 37%) have violations +1, while 38 (or a little bit more than 63%) have no violation of (14).

For the time being, we have some theories, but yet no convincing explanation for these peculiarities of the range $0° < \alpha \leq 10°$. But we want to stress that of the about 900 angles in the range $0° < \alpha \leq 30°$ where (14) is exactly fulfilled, 555 alone can be found among the first 10 degrees. Alternatively: of the about 1150 angles of the range $0° < \alpha \leq 45°$ where (14) is exactly fulfilled, almost half of these come from the first 10 degrees. This seems to indicate that Method II was strictly applied only for the first 10°, and this is exactly where Method I by having to use large multipliers (e.g. multiplier 4 from 10° down to 7°11', then multiplier 8 from 7°10' to 3°35', then multiplier 10 down to 2°52' etc.) is most error-prone. Termed differently, the percentage of angles exactly fulfilling (14) decreases sensibly, from 92.5% in the interval $0° < \alpha \leq 10°$ via 50% in the interval $0° < \alpha \leq 30°$ to 42.5% in the interval $0° < \alpha \leq 45°$.

The last of the three peculiarities just mentioned extends in some way to the range $10° < \alpha \leq 45°$. As we observed, starting right at 10°1', the number of violations per degree or even per 10' interval rises sharply. On the other hand, we still have a large number of *non-violating* multiples of 10', namely the following 119:

10°10', 10°20', 10°40', 10°50', 11°10', 11°20', 11°30', 11°50', 12°10', 12°20', 12°30', 13°10', 13°20', 13°30', 13°40', 13°50', 14°10', 14°30', 14°40', 14°50', 15°0', 15°10', 15°20', 15°30', 15°40', 15°50', 16°20', 16°30', 17°0', 17°10', 17°20', 17°50', 18°0', 18°10', 18°40', 19°0', 19°10', 19°20', 19°30', 19°40', 19°50', 20°0', 20°10', 20°20', 20°40', 20°50', 21°0', 21°10', 21°30', 21°40', 21°50', 22°0', 22°10', 22°20', 22°30', 22°50', 23°0', 23°10', 23°20', 23°30', 23°40', 24°0', 24°10', 24°40', 24°50', 25°20', 25°40', 25°50', 26°0', 26°10', 26°20', 26°30', 26°40', 26°50', 27°0', 27°10', 27°20', 27°40', 27°50', 28°0', 28°10', 28°20', 28°30', 28°40', 28°50', 29°0', 29°10', 29°20', 29°40', 30°0', 30°10', 30°50', 31°0', 32°10', 32°30', 32°40', 33°20', 33°40', 33°50', 34°20', 35°0', 35°30', 35°50', 38°20', 39°0', 39°30', 40°0', 40°30', 40°50', 41°0', 41°10', 41°30', 41°40', 42°20', 42°30', 43°20', 43°50', 44°10' and 44°20'.

As we have 210 multiples of 10' in the interval $10° < \alpha \leq 45°$, these 119 angles make for almost 57% of the non-violating multiples of 10', compared with a little bit more than 63% in the first 10° interval $0° < \alpha \leq 10°$.

V.

Investigating the range $0° < \alpha \leq 10°$ also helps to detect at least one severe computing error and its consequences as well as its origin. Looking at $\beta = 0°57'$ one finds LN(SIN(0°57')) = 41006643, both in Napier 1614 and Napier 1620. This is about 11000 units too big, but using LN(SIN(90°–0°57')) = LN(SIN(89°3')) = 1375, LN(SIN(1°54')) = 34076549 and LN($\frac{1}{2}$h) = 6931469, all taken from the 1614 table, we find that (14) becomes

$$\begin{aligned}&\mathrm{LN}(\mathrm{SIN}(\beta)) + \mathrm{LN}(\mathrm{SIN}(90°-\beta)) - \mathrm{LN}(\mathrm{SIN}(2\beta)) - \mathrm{LN}(\tfrac{1}{2}\mathrm{h}) =\\ &41006643 \ + \ 1375 \ - \ 34076549 \ - \ 6931469 = 0,\end{aligned}$$

i.e. (14) is fulfilled – which of course it should not be, because 41006643 is severely wrong. But if we take a further look at LN(SIN(γ)) : = LN(SIN(2β)) = LN(SIN(1°54')) = 34076549, then here we find (14) violated: Taking LN(SIN(90°–1°54')) = LN(SIN(88°6')) = 5499, LN(SIN(3°48')) = 27139185 and again LN($\frac{1}{2}$h) = 6931469, all from the 1614 table, we get

$$\begin{aligned}&\mathrm{LN}(\mathrm{SIN}(\gamma)) + \mathrm{LN}(\mathrm{SIN}(90°-\gamma)) - \mathrm{LN}(\mathrm{SIN}(2\gamma)) - \mathrm{LN}(\tfrac{1}{2}\mathrm{h}) =\\ &34076549 \ + \ 5499 \ - \ 27139185 \ - \ 6931469 = 10998.\end{aligned}$$

By observing that $10998 = 2 \cdot 5499$, we immediately see what happened: As Napier used

$$\mathrm{LN}(\mathrm{SIN}(\gamma)) = \mathrm{LN}(\mathrm{SIN}(2\gamma)) - \mathrm{LN}(\mathrm{SIN}(90°-\gamma)) + \mathrm{LN}(\tfrac{1}{2}\mathrm{h}),$$

he should have *subtracted* LN(SIN(90°–γ)) = LN(SIN(88°6')) = 5499 on the right hand side, but erroneously *added* it instead, thus making LN(SIN(1°54')) $2 \cdot 5499 = 10998$ units too big. By using Method II, this error was transferred to $\beta = \gamma/2$, i.e. to LN(SIN(0°57')), but because in Method II the LN(SIN(..)) of an angle with an odd number of minutes is not used again, once it has been computed (see III.), this error doesn't propagate further. However, that's not all:

Between 1614 and 1620, someone must have stumbled across this very erroneous value of LN(SIN(1°54')). The 1620 Lyon edition of the *Descriptio* was clearly set anew from a copy of the 1614 edition, thereby inevitably introducing some new errors. But at 1°54' we surprisingly find LN(SIN(1°54')) = 34065549, exactly 11000 units less than in the 1614 edition – which makes this value almost OK. Whoever made this correction, however, quite obviously did not make use of Method II, for in this case he would have obtained 34065551. And not using (or disrespecting) Method II, he also wasn't able to recognise that this correction of the value of LN(SIN(1°54')) might also affect LN(SIN(0°57')), as it is the case. So in Napier 1620 we find that LN(SIN(1°54')) has been corrected in an acceptable way, but LN(SIN(0°57')) has been left uncorrected and thus in both editions is 41006643 (instead of 40995645 according to Method II when using the correct value LN(SIN(1°54')) = 34065551).

VI.

As pointed out above, there is some evidence that Napier used Method II *exactly* from 10° downward. This, together with the many angles having a multiple of 10' combined with a peculiar property, might lead us to suspect that Napier could have followed a system with three or four levels in the course of his computations. It may have been that he first computed LN(SIN(90°0')), LN(SIN(80°0')), LN(SIN(70°0'))…. LN(SIN(30°0')) with his interpolation procedure for the starting interval, and then LN(SIN(20°0')) and LN(SIN(10°0')) with Methods I or II. This would have given him a quick overview of the quantitative development of LN. All these SINes have been correctly rounded – with the exception of 50°0', which is one unit too big – and have the same values in Reinhold, Finck, Lansbergen and Napier. The values given for LN(SIN(90°0')), LN(SIN(80°0')), LN(SIN(70°0'))... LN(SIN(30°0')) have been computed without error, when compared with a re-computation using Napier's erroneous values for LN(9995000) and LN(9900000). So let's have a look at LN(SIN(20°0')) and LN(SIN(10°0')) and see whether Napier's values can be obtained by Method I or by Method II.

We have SIN(20°0') = 3420201; the required multiplier in Method I is 2, so we next need LN(6840402), which – by the interpolation procedure for the starting interval – is 3797385; adding LN($\frac{1}{2}$h) = 6931469 we obtain 10728854, but in the table we find 10728852 – which, however, is the value we get by applying Method II. On the other hand we have SIN(10°0') = 1736482; the required multiplier is $4 = 2^2$, so we next need LN(6945928), which – again by the interpolation procedure for the starting interval – is 3644294; adding 2 · LN($\frac{1}{2}$h) = 13862938 we obtain 17507232, but in the table we find 17507234. Method II in this case would result in 17507233, so in this case *both* methods do *not* exactly reproduce Napier's value, but his value is nearer to the one obtained by Method II.

A next step would perhaps be to look at the values for angles being a multiple of 5°, or directly going to whole degrees; and the same procedure could have been pursued for the subdivision into minutes: first to compute the 10' multiples, then refining to steps of 1'. But this is mere speculation for the time being.

One thing should be mentioned here, although this topic, too, has not yet been the object of detailed studies. Even if 92.5% of the angles in the first 10° (or 555 out of 600) have their LN values obeying (14), this does *not* imply that these LN values are correct (for LN(x) + c would also give correct results). But the use of Method II, confirmed by the many of Napier's values obeying (14), "draws" his LNs nearer to the true LN values, because (11b) or (14) now assume the character of a functional equation *defining* LN(SIN(..))! This is mirrored again by two comparisons: Considering the starting interval, we have a very good agreement of Napier's LN values with the true ones scaled down by 0.999999627, both based on the rounded SIN(α) values available to Napier. However, doing this for the interval $0° < \alpha \leq 10°$, this is no longer the case; we find big and bigger deviations. But we get deviations of only a *few* units when we compare Napier's values with the LN(h · sin(α)) values (of course also scaled down accordingly), when not SIN values, but *true* sin values (not available to Napier) are used. But these few units (in general less than 4) of deviation are most probably nothing but the starting errors introduced by having to use previously computed LN(SIN(α)) and LN(SIN(90°–α)) values…

VII.

Let's return a last time to the comparison between Methods I and II and the possible point of change. As long as Napier was using Method I in the first interval of extension, i.e. in [$\frac{1}{4}$h, $\frac{1}{2}$h), corresponding to k = 1 (see above), his multiplier was 2, and he must have used (7b), LN(x) = LN(2x) + LN($\frac{1}{2}$h), for his computations. We have SIN(14°29') = 2500984 > 2500000 = h/4, so the first step of Method I would lead him from 30° down to 14°29'. We can estimate the error at that point: As SIN(2 · 14°29') ≈ SIN(29°) ≈ SIN(30°), we will have the error of SIN(30°) = $\frac{1}{2}$h (which is –3) doubled according to (7b); in numbers: –6. This is exactly what we find in Napier's LN(SIN(14°29')) = 13859004, compared to –h · ln(sin(14°29')) = 13859010. However, Napier obviously was not in a position to observe this; instead, initially he relied on his forward analysis only. So what might have made him think about errors again?

As we said, Napier in general gives one example (but rarely more than one) for his propositions. This goes for the *Constructio* as well as for the *Descriptio*. Now in the latter we find one example, which *en passant* was already quoted above: Book I, Chapter V, Problem 3, which illustrates how to solve proportions with the aid of LN: Let's have a : b = c : x, and x being sought. Once again (5) at once tells us how to do this: LN(a) – LN(b) = LN(c) – LN(x), and therefore simply LN(x) = LN(b) + LN(c) – LN(a); after looking up x from the LN table we're done. In Napier's illustrating example we have a = 7660445, b = 9848078, c = 5000000. As Napier says: *Hoc vulgus acquirit ducendo secundum in tertium, & dividendo per primum* (freely translated: Usually this is done by multiplying the second term with the third term and dividing the product by the first term). This of course would mean a lot of calculating, even if in this particular case the third term makes it somewhat easier; whereas using LNs it is replaced by three lookups, one addition, one subtraction and one final lookup.

But "the proof is in the pudding": The numbers given in this example are a = SIN(50°), b = SIN(80°) and c = SIN(30°); looking up we find their LNs as LN(a) = 2665149, LN(b) = 153088 and LN(c) = 6931469, so LN(x) = 153088 + 6931469 – 2665149 = 4419408, and looking up again we find x = SIN(40°). Napier ends: *Idem proveniret si (spretis sinibus) solum darentur tres sui arcus 50. gra. 80. gr. & 30. gr. Namque ex Logarithmis arcuum 80. gr. & 30. gr. ablato Logarithmo 50. gr. remanebit Logarithmus 40. gr. Et ita ipse arcus 40. gr. innotescet absque sinibus, eorumve multiplicatione aut divisione, prout initio polliciti sumus* (So it results (not using sines) that given only the arcs of 50, 80 and 30 degrees, we obtain the logarithm for [the sine of] 40 degrees by subtracting the logarithm for [the sine of] 50 degrees from the sum of the logarithms for [the sines of] 50 and 80 degrees, without using, multiplying or dividing their sines, as we above promised we could do).

Three (or even four) facts seem remarkable in this example: First, this is an instance of (10b) and/or (11a), i.e. Method II. For indeed we have SIN(50°) : SIN(80°) = SIN(30°) : SIN(40°), according to Napier's identity (10b). Moreover, second, it is shown for a particular case where only angles appear which are an exact multiple of 10°. Third, we can see that Napier here in the *Descriptio* uses his identity (10b) to give a perhaps surprising example with "round" angles, while in the *Constructio* he *proved* this relation for the SINes of any angles. So it might well be that he looked for such simple relations in order to check whether his LN table, perhaps already computed right to the end with Method I, leads to correct results, and that doing this for relations in triangles with at least one very small angle he observed that this was not the case (there are several such small-angled triangles in the *Descriptio's* examples). This, then, in turn might have given him the idea to use his observation the other way round, i.e. to derive a means for *computing* LNs for small angles. (The fourth remarkable fact is that Napier here explicitly *tells* us which angles are involved *and* that their SINes do not have to be used – such hints and remarks are usually suppressed or very well hidden…)

VIII.

For us it seems too early to give a definite and final answer to the question "Which SIN table did Napier use?" Maybe one really has to distinguish between *use* and *print*, at least in some parts of the table. We do accept the findings of Glowatzki & Göttsche, but not necessarily their conclusion that Napier must have used either Finck's or Lansbergen's table. To make our differing interpretation clear, here is an example:

Napier has a simple (and correct) rule for computing LN(SIN(α)) if α is in the range from about 88°36' to 90°0'; this rule is simply LN(SIN(α)) = h – SIN(α). Among the 85 angles in this range, we have 24 (corresponding to a remarkable 30%) where Reinhold's SIN values differ from Finck's/Lansbergen's/Napier's SIN values. But in *all* of these 85 cases, it is *Reinhold's* SIN value which has been *used* in computing LN(SIN(α)) = h – SIN(α), although Finck's (or Lansbergen's) SIN value has been *printed*… This is very easy to check for everyone even without the aid of a computer by simply looking at the last figures of SIN(α) and LN(SIN(α)): As LN(SIN(α)) = h – SIN(α) is equivalent to LN(SIN(α)) + SIN(α) = h = 10000000, the last figures of LN(SIN(α)) and SIN(α) should add up to 0 (mod 10). But in as many as 24 cases they *don't add up* to 0 (mod 10). And because *all* of Reinhold's wrongly rounded SINes in this 85' interval are one unit too small, but 7 of the 24 "corrected" values of Finck and Lansbergen have the same error in the other direction, i.e. are one unit too big, we are even lead to 7 differences of +2 (and consequently 17 remaining differences of +1), when we compare the sum of the *printed* values of SIN(α) and LN(SIN(α)) with the ideal result 10000000, as long as α is in the range from 88°36' to 90°0'. Using Reinhold's values, however, we always obtain 10000000; but Reinhold's values do *not* appear in print…

*

5. Conclusion

We have led our investigations I. through VIII. to different levels of depth, according to the possibilities either in the material or in our own limitations. What can be said definitely, however, is that even after 400 years it is worthwhile and extremely rewarding to have a look at the *Constructio*, because it reveals, when read anew, many details that have been hidden by the Euclidean style of the original. A first approach was given already in Fischer 1997 and Fischer 1998; this article means to proceed in the same direction, i.e. concentrates on the numerical data in combination with the rules and formulas given by Napier. We presented this work for the first time in the Mini-Workshop *History of Numerical and Graphical Tables* in

Oberwolfach, February 27th – March 5th, 2011. Unfortunately, personal developments and circumstances forced both of us to temporarily abandon the matter; nonetheless, we believe the full publication in the state we had to leave it was necessary, and we hope that some day either we or other researchers will find the time to resume this work.

*

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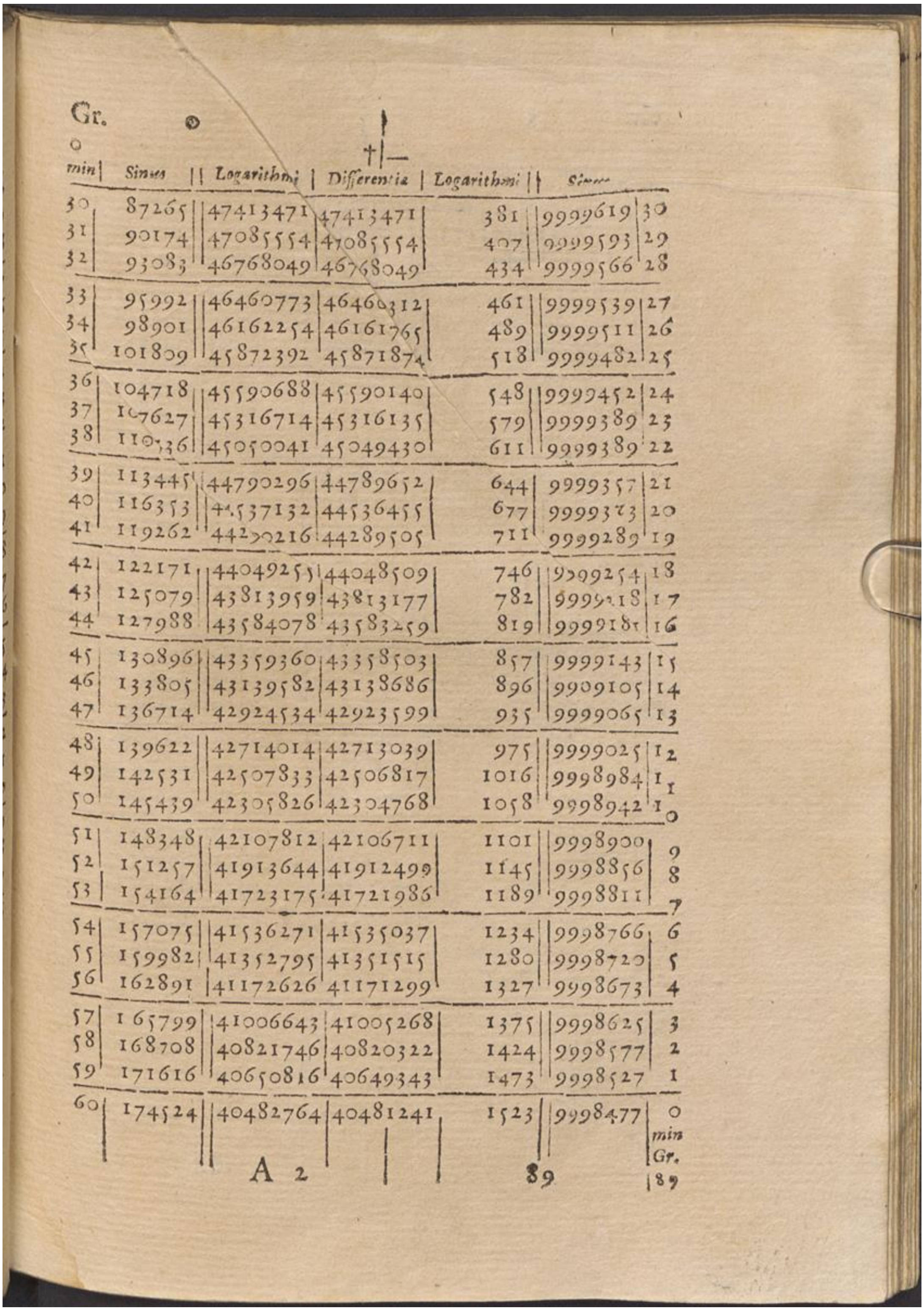
Gr. 0

min	Sinus.	Logarithmi	Differentiæ	logarithmi	Sinus	
0	0	Infinitum	Infinitum	0	10000000	60
1	2909	81425681	81425680	1	10000000	59
2	5818	74494213	74494211	2	9999998	58
3	8727	70439564	70439560	4	9999996	57
4	11636	67562746	67562739	7	9999993	56
5	14544	65331315	65331304	11	9999989	55
6	17453	63508099	63508083	16	9999986	54
7	20362	61966595	61966573	22	9999980	53
8	23271	60631284	60631256	28	9999974	52
9	26180	59453453	59453418	35	9999967	51
10	29088	58399857	58399814	43	9999959	50
11	31997	57446759	57446707	52	9999950	49
12	34906	56576646	56576584	62	9999940	48
13	37815	55776222	55776149	73	9999928	47
14	40724	55035148	55035064	84	9999917	46
15	43632	54345225	54345129	96	9999905	45
16	46541	53699843	53699734	109	9999892	44
17	49450	53093600	53093577	123	9999878	43
18	52359	52522019	52521881	138	9999863	42
19	55268	51981356	51981202	154	9999847	41
20	58177	51468431	51468361	170	9999831	40
21	61086	50980537	50980450	187	9999813	39
22	63995	50515342	50515137	205	9999795	38
23	66904	50070827	50070603	224	9999776	37
24	69813	49645239	49644995	244	9999756	36
25	72721	49237030	49236765	265	9999736	35
26	75630	48844826	48844539	287	9999714	34
27	78539	48467431	48467122	309	9999692	33
28	81448	48103763	48103431	332	9999668	32
29	84357	47752859	47752503	356	9999644	31
30	87265	47413852	47413471	381	9999619	30

89

Napier's *Descriptio*, 1614 edition, first page of the LN table. Within the 7-figure precision, SIN(89°59') and SIN(90°0') are identical and = 10000000, so SIN(89°59') and LN(SIN(89°59')) add up to 10000001, whereas Reinhold's table has SIN(89°59') = 9999999, so with that value the correct result 10000000 would appear. Observe that also in several other cases SINes and their respective logarithms do not add up to 10000000, but to 10000001 or 10000002; in particular the 7 successive SINes from SIN(89°48') to SIN(89°54') given here differ from Reinhold's values by 2 units, and accordingly SIN and LN add up to 10000002 instead of 10000000 (see VIII.).



Gr. 0 min	Sinus	Logarithmi	Differentia	Logarithmi	Sinus	
30	87265	47413471	47413471	381	9999619	30
31	90174	47085554	47085554	407	9999593	29
32	93083	46768049	46768049	434	9999566	28
33	95992	46460773	46460312	461	9999539	27
34	98901	46162254	46161765	489	9999511	26
35	101809	45872392	45871874	518	9999482	25
36	104718	45590688	45590140	548	9999452	24
37	107627	45316714	45316135	579	9999389	23
38	110536	45050041	45049430	611	9999389	22
39	113445	44790296	44789652	644	9999357	21
40	116353	44537132	44536455	677	9999323	20
41	119262	44290216	44289505	711	9999289	19
42	122171	44049255	44048509	746	9999254	18
43	125079	43813959	43813177	782	9999218	17
44	127988	43584078	43583259	819	9999181	16
45	130896	43359360	43358503	857	9999143	15
46	133805	43139582	43138686	896	9909105	14
47	136714	42924534	42923599	935	9999065	13
48	139622	42714014	42713039	975	9999025	12
49	142531	42507833	42506817	1016	9998984	11
50	145439	42305826	42304768	1058	9998942	10
51	148348	42107812	42106711	1101	9998900	9
52	151257	41913644	41912499	1145	9998856	8
53	154164	41723175	41721986	1189	9998811	7
54	157075	41536271	41535037	1234	9998766	6
55	159982	41352795	41351515	1280	9998720	5
56	162891	41172626	41171299	1327	9998673	4
57	165799	41006643	41005268	1375	9998625	3
58	168708	40821746	40820322	1424	9998577	2
59	171616	40650816	40649343	1473	9998527	1
60	174524	40482764	40481241	1523	9998477	0 min Gr. 89

A 2 89

Napier's *Descriptio*, 1620 edition, second page of the LN table. Observe that SIN(89°22') and SIN(89°23') are identical (due to a printer's error; Napier 1614 is OK here); the latter should read 9999421. Apart from this, SIN(89°2'), SIN(89°8'), SIN(89°9'), SIN(89°14'), SIN(89°21') and their respective logarithms do not add up to 10000000, but to 10000001 (see VIII.). Observe also LN(SIN(0°57')) = 41006643 (mentioned in V.), which should be 40995645.