Napier's main application: spherical trigonometry

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Abstract

The present article is meant as a historically and mathematically self-contained introduction to Napier's contribution to spherical trigonometry. Whereas his logarithms have been highlighted by many modern expositors, his trigonometric results have been abandoned to some degree. Napier himself, however, considered trigonometry as the main field of application of logarithmic computations. Although much of our material stems from Braunmühl's *Geschichte der Trigonometrie*, we also present some new or scarcely known details.

Introduction

John Napier's name is commonly connected with the invention of logarithms, a merit he shares with Jost Bürgi. He is also regarded as a pioneer of the concept of mathematical functions. His role in these two fields has been highly valued in many publications — in particular, in the contributions to the present volume, of course. Napier's logarithms totally overshadow his achievements in spherical trigonometry. Napier himself, however, considered trigonometric problems as the main application of his logarithmic method. This becomes evident by the fact that he published his trigonometric results together with his theory of logarithms, for instance, in his *Constructio* (see [8]).

It seems that the only complete survey of his trigonometric contributions can be found in the second volume of Braunmühl's *Geschichte der Trigonometrie* [3] of 1903. Besides the disadvantage of being written in German, Braunmühl's work takes it for given that the reader is well versed in spherical trigonometry. This is, of course, coherent with the important role of this matter in the higher education of the beginning twentieth century. The chronological structure of Braunmühl's exposition also implies that the discussion of important forerunners of Napier (such as Regiomontanus) is scattered over a number of places.

The aim of this article is a reasonably self-contained presentation of Napier's trigonometric work. Although we are deeply indebted to Braunmühl, we hope to contribute some new or scarcely known details, for instance, a remarkable flaw in Napier's *Constructio* (see Section 9).

Our discussion is based on the understanding of spherical trigonometry that was common in Napier's time. Hence it is not completely systematic and not without gaps from a modern viewpoint. For instance, some problems would require a distinction between acute and obtuse triangles — which we omit, restricting ourselves to a typical case. This restriction becomes most evident in Sections 6 and 7.

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1. Spherical trigonometry

Up to the age of the Renaissance trigonometry has been, almost exclusively, an auxiliary science of astronomy. So its natural frame is the celestial sphere, whose center is the earth. The latter can be imagined as a tiny ball, almost a point only. The celestial sphere revolves around the axis formed by the line through the north and the south poles. One revolution lasts about 24 hours. If we connect three points on the said sphere by arcs of great circles, we obtain a spherical triangle. A typical example of such a triangle has its corner points in the north pole A, the zenith B of an observer on the northern hemisphere and a third point C which marks the position of some star under observation. The meridian arcs c (meridian of the observer) and b (meridian of the star in question) form, together with the arc a of the great circle connecting B and C, the *sides* of the triangle. These sides have to be understood as the *angles* under which the terrestrial observer sees A, B, and C. Suppose, for instance, the geographic latitude of the observer is 29 degrees. Then $c = 90^\circ - 29^\circ = 61^\circ$.

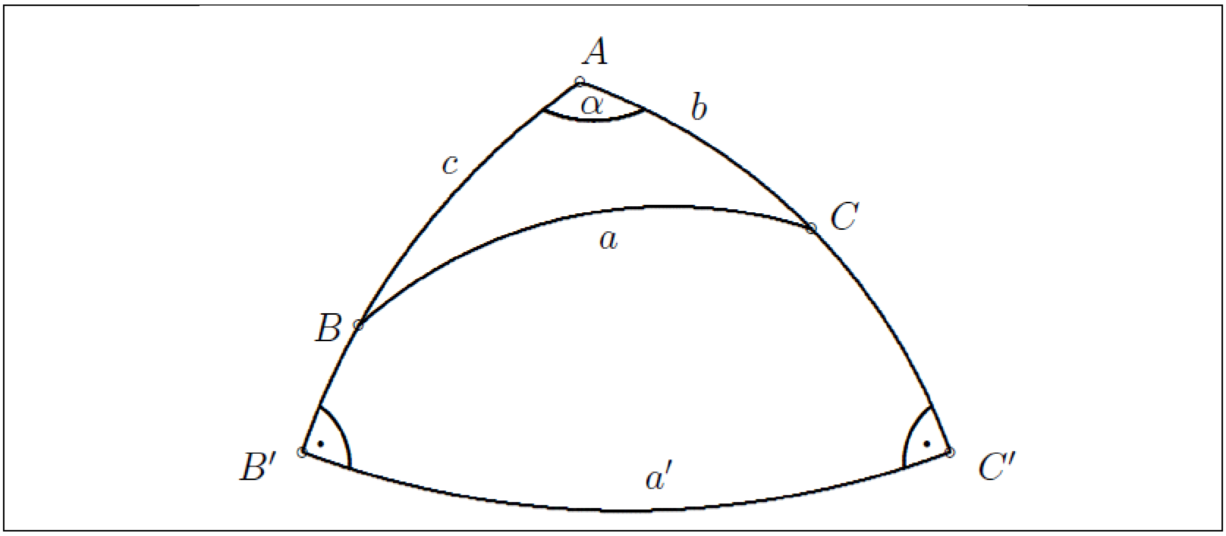

Figure 1

In addition, a spherical triangle also has *angles* in the proper sense of the word. In Figure 1, we obtain the angle α at the corner A on extending the side c to 90° from A to B' and, in the same way, the side b to 90° from A to C'. The angle that measures the side a' of the triangle $AB'C'$ (this side is a part of the celestial equator) is the angle α in question. It could also be defined as the angle formed by the tangents of the meridians in A. Hence we may write $a' = \alpha$.

The fundamental theorem of spherical trigonometry is the *cosine rule*, which takes two different forms (see [2] or [14], p. 18 ff.). The *cosine rule for sides* reads

$$\cos a = \cos b \cos c + \sin b \sin c \cos \alpha. \qquad (1)$$

Accordingly, our observer can find the angle α provided that he has measured the sides a, b, and c. Thereby, he knows the difference in longitude between his own position and the point on the earth situated below the star C. If the star's position at a given time is known, the longitude of the observer can be determined.

Of course, the quantities a,b,c and α do not play a distinguished role. The identity (1) remains valid under a cyclic permutation of the sides a,b,c and, simultaneously, the angles α,β,γ. This identity, however, renders more than the computation of an angle from the sides of the triangle. We can also compute a side of the triangle from the two remaining sides and the angle included by them (for example, a from b, c and α). Even the case of two given sides and an angle opposed to one of these sides can be treated by means of (1). Suppose we know a,b, and α. Then (1) leads to a quadratic equation for $\cos c$, if we use

$$\sin c = \sqrt{1 - \cos^2 c} \tag{2}$$

At this point one should observe that all sides and all angles of the triangle take values strictly between 0° and 180°. Thus, $\sin c$ is always positive, and so it suffices to take the *positive* root in (2). The resulting quadratic equation may have two geometrically meaningful solutions, occasionally, however, none.

In contrast with plane triangles, the sum $\alpha + \beta + \gamma$ in a spherical triangle is always greater than 180° (and less than 540°, since each angle is less than 180°). This has the effect that a spherical triangle is completely given by its angles. The *cosine rule for angles*

$$\cos\alpha = -\cos\beta\cos\gamma + \sin\beta\sin\gamma\cos a \tag{3}$$

yields the solution of the remaining cases of a spherical triangle, i.e., three given angles or two angles and the side adjacent to both of them. The case of two angles and a side opposed to one of them again leads to a quadratic equation, with the aid of (2).

A proof of the cosine rule can be found in the pretty booklet [14] or in more extensive works like [12] and [13]. In Section 6 we shall see that both versions (1) and (3) of this rule are equivalent in an elementary way.

2. Napier's preconditions

Spherical trigonometry is a basic tool in the *Almagest*, Ptolemy's great astronomical handbook (about 150 AD). But in antiquity its appearance differs very much from that of modern times (see [10], p. 64 ff.). Greek mathematics does not work with formulas, but phrases its results in words, often in the form of rather involved periods. This also holds for proofs, of course. Hence it is quite important — also for a modern student of Greek mathematics — that every theorem and every proof is accompanied by drawings. In addition, the Greeks have only one alphabet, namely that of Greek capital letters, to denote mathematical objects. Accordingly, points are denoted by capital letters, line segments (or segments of circular arcs) by pairs of letters denoting their endpoints.

This tradition is dominant well into the seventeenth century; Napier's main works (published 1614 and 1619) are no exceptions from this rule.

For our purpose it may be instructive to see a rather literal translation of the cosine rule (1) for sides from Regiomontanus' book "De triangulis omnimodis" (see [11], p. 127; first printed in 1533, almost 60 years after the death of its author).

In every spherical triangle consisting of great circular arcs, the ratio of the versed sine of an arbitrary angle to the difference of two other versed sines, one of which belongs to the side that subtends that angle, whereas the other belongs to the difference of the arcs attached to this angle, equals the ratio of the square of the sine of the right angle to the rectangle formed by the sines of the arcs attached to the said angle.

Regiomontanus illustrates this theorem by a drawing. We use Figure 1 for this purpose. *Sine* in Regiomontanus' sense (also called the right sine) means R·sin in our sense, where R is the (rather large) radius of the sphere in question. Accordingly, the sine of the right angle equals R. The *versed sine* stands for R·(1−cos). Hence Regiomontanus says that the following identity of ratios holds:

$$R(1 - \cos \alpha) : (R(1 - \cos a) - R(1 - \cos(b - c))) = R^2 : (R \sin b \cdot R \sin c) \qquad (4)$$

If we cancel the radii R on both sides and simplify the difference in the denominator on the left hand side, we obtain

$$(1 - \cos \alpha) : (\cos(b - c) - \cos a) = 1 : (\sin b \ \sin c),$$

which is equivalent to

$$\sin b \sin c - \sin b \sin c \cos \alpha = \cos(b - c) - \cos a. \qquad (5)$$

By means of the identity $\cos(b - c) = \cos b \cos c + \sin b \sin c$, formula (5) is transformed into (1).

Regiomontanus applies his theorem to the case when three sides are given and one angle shall be computed. He also indicates that, if three of the four quantities a,b,c,α are known, the remaining one can be computed by means of (4). In Regiomontanus' case of three given sides, formula (5) yields

$$\text{sinvers}\ \alpha = \frac{\text{sinvers}\ a - \text{sinvers}\ (b - c)}{\sin b \sin c}, \qquad (6)$$

where we have written, following Regiomontanus, sinvers instead of 1 − cos.

Evaluating the right hand side of (6) requires subtractions, but also one multiplication and one division. The latter operations are rather toilsome if multi-digit numbers are involved (as is the case with astronomical computations). In Napier's time astronomers would have used the identity

$$\sin a \sin b = (\cos(a - b) - \cos(a + b))/2, \qquad (7)$$

for the denominator of (6). Thereby, the multiplication is replaced by an addition, a subtraction, and a bisection. This kind of simplification was called *prosthaphairesis* (a term going back to the almagest), which can be translated by “addition and subtraction”.

One should be aware of the fact that prosthaphaeresis was not a trivial process for Napier’s contemporaries, since they had no formulas at hand. In particular, the identity (7) used here was disposable only in the vestment of a clumsy theorem (see [3], vol. 1, p. 197 ff.).

3. Logarithms and trigonometry

In order to simplify calculations of the kind just discussed, John Napier invented his *logarithms* — the name also goes back to him. He published his system of logarithms in two booklets, whose titles were *Mirifici logarithmorum canonis descriptio* (1614) and *Mirifici logarithmorum canonis constructio* (which appeared in 1619 when Napier was already dead). Both tracts together were republished in 1620, see [8]. This edition is generally accessible in digitized form (see references). Therefore, we always use the edition [8] and quote Napier’s tracts under the names of *Descriptio* and *Constructio*.

Napier’s tracts also contain his results in trigonometry. As we mentioned above, Napier considered trigonometry as the main field of application of his logarithms. This explains why his results in both fields were published together.

Napier’s objective is continuous logarithmic computation in spherical trigonometry. To this end he has to reshape the respective formulas in such a way that they consist only of *products* and *quotients* of values of trigonometric functions — but not of sums or differences of these values. Then the application of logarithms reduces the computation to *additions* and *subtractions*. But we have to bear in mind that formulas belong to *our* mathematical world; Napier did not have this tool at his disposal. It does not mean a derogation of his ideas, if we replace his somewhat unwieldy logarithm (see [1], [6]) by the *decadic* logarithm log, which was introduced by Henry Briggs (1561–1630) shortly after. Accordingly, $\log 1 = 0$, $\log 10 = 1$, $\log 100 = 2$ and so on. In addition, the radius R of the foregoing section will always be 1.

Braunmühl rightly says that Napier undertook a complete reorganization of spherical trigonometry (see [3], vol. 2, p. 12). This reorganization, however, suffers from the fact that Napier also feels indebted to the users of prosthaphaeresis, with the effect that he presents logarithmic theorems together with equivalent prosthaphaeretic results — a feature that becomes most obvious in the *Constructio*. This fact detracts from Napier’s main objective, although it is quite understandable against the background of his time.

4. Napier’s rules

In book II, chapter IV, of the *Descriptio* Napier formulates the rules for a right spherical triangle which bear his name. Let ABC be a spherical triangle with a right angle at C. The natural determiners of this triangle are (if we omit $\gamma = 90°$) a, b, α, c, β, in this order. Two neighbouring determiners in this series are called *adjacent*; and finally, β and a are also considered as adjacent.

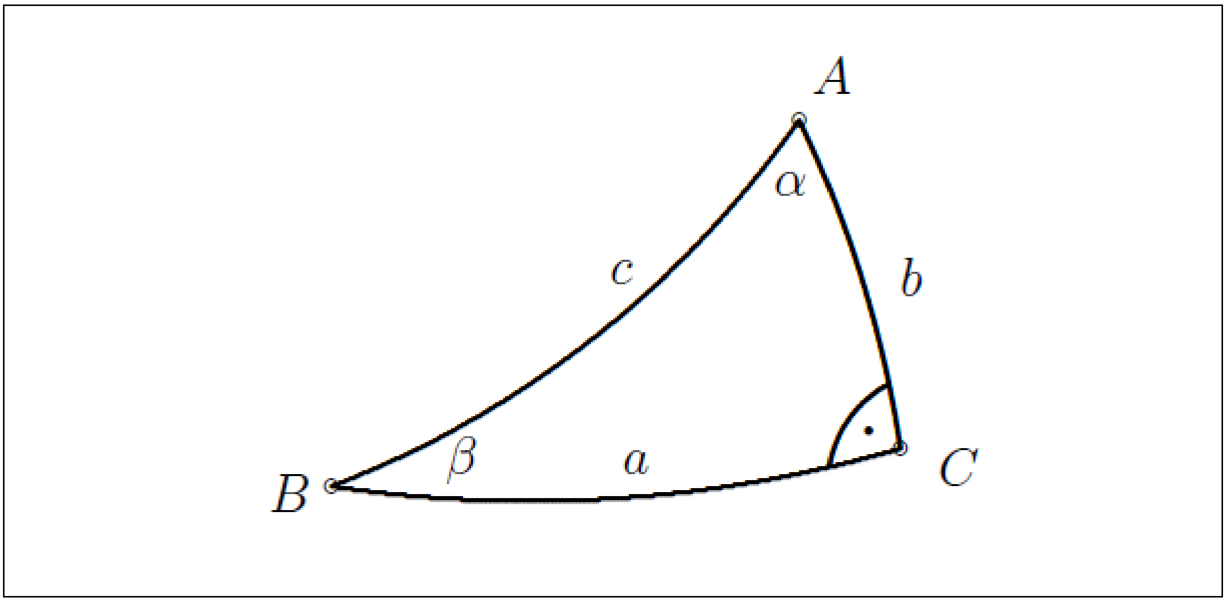

Figure 2

The legs a and b of the right triangle are now replaced by their complements $\bar{a} = 90^\circ - a$, $\bar{b} = 90^\circ - b$. Then $\bar{a}, \bar{b}, \alpha, c, \beta$ are called the *circular parts* (Napier says "partes circulares") of the triangle. Now Napier's rules can be phrased as follows (see [8], *Descriptio*, p. 33):

(a) *The cosine of each (circular) part equals the product of the cotangents of its adjacent parts.*

(b) *This cosine is also equal to the product of the sines of the non-adjacent parts.*

For instance, let us consider c, whose non-adjacent parts are $\bar{a}$ and $\bar{b}$. By rule (b), $\cos c = \sin \bar{a} \sin \bar{b}$, i.e.,

$$\cos c = \cos a \cos b. \tag{8}$$

This becomes obvious from the cosine rule (1), since $\gamma = 90^\circ$ and, thus, $\sin a \sin b \cos \gamma = 0$. In the same way rule (a) for the adjacent parts α and β of c reads

$$\cos c = \cot \alpha \cot \beta; \tag{9}$$

here the cosine rule (3) gives $\cos \gamma = 0 = -\cos \alpha \cos \beta + \sin \alpha \sin \beta \cos c$; accordingly, division by $\sin \alpha \sin \beta$ yields (9). Napier's rules say that (8) and (9) remain valid, if the circular parts undergo a cyclic permutation. In this way (8) gives $\cos \beta = \sin \alpha \cos b$.

Formulas of this kind are well suited for logarithmic computation. For example, if one has read logtan α and logtan β from a table, −logcos c is just the sum of these numbers, by (9). Then the table yields c.

All of the ten formulas that arise from (8) and (9) by cyclic permutation of the circular parts were known before Napier. This fact might give the impression that Napier's rules are only *mnemonic* inventions that reduce ten formulas to two. This, however, is not true, as has been put straight in [7] and [2], p. 19 ff. Indeed, these rules result from a beautiful geometric configuration, which we shall present now.

For this purpose we need an additional notion of spherical trigonometry, which we explain by Figure 1. Let *A* be an arbitrary point on the celestial sphere. We extend the (arbitrarily chosen) arcs *b* and *c* of great circles through *A* both to a length of 90°. Then the arc $B'C'$ lies in a plane through the midpoint *M* of the sphere, which is at right angle to the line *AM*. The great circle lying in this plane is called the *polar* of *A*. If *A* is the north pole, say, its polar is the (celestial) equator. The angles at B' and C' are both equal to 90°. This can be seen from the example of the equator since each meridian intersects the equator at right angles. The polar of *A* may also be characterized by these right angles of intersection. Indeed, it is the great circle which is intersected by two great circles through *A* (which may be chosen arbitrarily) at right angles.

In what follows we use Napier's own configuration (see [8], *Descriptio*, p. 32), although we slightly modify his arguments.

Since the earth is tiny relative to the celestial sphere, we may assume that the plane of the horizon goes through the center *M* of this sphere. This plane intersects the sphere in the *circle of the horizon*. Let *BS* be an arc of this circle and assume that the observer is in *B* and the sun in *S*. The meridian *BP* of the observer goes through the north pole *P*. The angle at *B* amounts to 90°. The arc *PS* is the meridian of the sun. If we extend the meridian *PB* beyond *P* by an arc of 90°, we arrive at the point *D* on the celestial equator. On the other hand, if we extend the meridian *PS* beyond *S* to a total of 90°, we arrive at the point *F*, which also lies on the equator. Hence the arc *DF* is part of the equator. By what we said above, the angles at *D* and *F* are right ones. What we have done for the point *P*, we can also do for *S*. We extend *BS* beyond *S* by a total of 90° and complement *SP* beyond *P* up to 90°. The arc *CE* that arises in this way is part of the polar of *S*, and the angles at *C* and *E* are again right ones. Altogether, we obtain a spherical pentagon *OQZPS* with right spherical triangles adjacent to its sides.

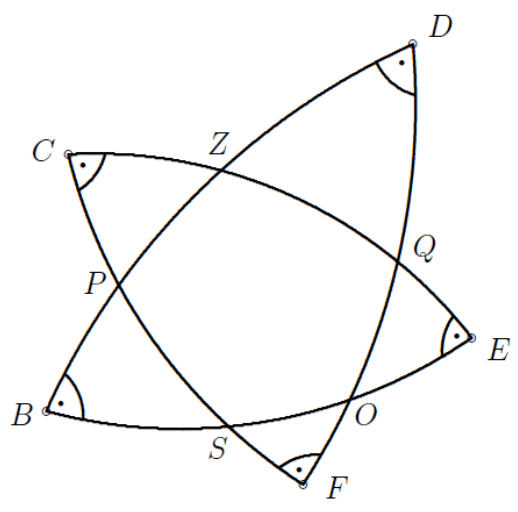

Figure 3

It turns out that the circular parts of these five triangles are identical — up to cyclic permutations. We show this in the case of the triangles *BSP* and *CPZ* in an exemplary way. In the case of *BSP* these parts are

$$90^\circ - PB,\ 90^\circ - BS,\ \angle S,\ PS,\ \angle P. \tag{10}$$

For the triangle *CPZ* we have, accordingly,

$$90^\circ - ZC,\ 90^\circ - CP,\ \angle P,\ PZ,\ \angle Z. \tag{11}$$

The angle $\angle P$ is an obvious common part. Its "predecessor" in the triangle *CPZ*, namely, $90° - CP$, equals *PS*, since *C* arose by complementing *PS* to 90° . The "successor" of $\angle P$ in the second triangle is PZ. But BE is part of the polar of Z, since the angles at B and E are right ones — which characterizes the polar. Therefore, $ZB = 90^\circ$ and, thus, $PZ = 90^\circ - PB$. Since *BE* is part of the polar of *Z*, the *interior angle* of the pentagon at the corner *Z* equals *BE*, as we explained in Section 1 by means of Figure 1. This interior angle amounts to $180^\circ - \angle Z$; and since SE (the extension of *BS*) is 90°, we have $90° - \angle Z = BS$, i.e., $\angle Z = 90 - BS$. With similar considerations one shows that the interior angle at S of the pentagon is $90 - ZC = \angle S$.

Accordingly, we have shown that the sequence (11) of circular parts has the form

$$\angle S,\ PS,\ \angle P,\ 90^\circ - PB,\ 90^\circ - BS.$$

Hence the terms of the sequence (10) have been shifted to the right just by two positions. If we apply (8) — phrased for circular parts — to the hypotenuse of the first triangle, we obtain

$$\cos(PS) = \sin(90^\circ - PB)\sin(90^\circ - BS).$$

The same procedure yields, for the second triangle,

$$\cos(90^\circ - PB) = \sin(\angle S)\sin(PS).$$

This, however, can be read as a statement about the first triangle in the sense of Napier's rule (b).

Of course, the two triangles just considered do not play a distinguished role, the geometry being the same for each other triangle. For instance, the arc *CF* is part of the polar of *Q*. This means that all of the five triangles have *the same* circular parts, however, in shifted positions. For instance, in the triangle *DZQ* these parts are shifted to the left by two positions, relative to those of *CPZ*. Therefore, each of (8) and (9) yields five formulas — which can be replaced by the rules (a) and (b).

The diagram of Figure 3 was called *Pentagramma mirificum* by Gauss, i.e., "miraculous pentagram" (see [3], vol. 2, p. 12, footnote 4). However, it is not true that Napier's rules have been theoretically established only by Gauss, as [13], p. 284, says. In fact, Napier's idea (which had its predecessors, as can be seen from the said p. 12) remarkably connects geometrical intuition with practical organization. By the way, Napier enunciates his rules in the language of logarithms, as we said in connection with

(9). He is well aware of the practical value of his achievements, since he says that his rules help avoiding the confusion that arises from the natural determiners and their rules (see [8], *Descriptio*, p. 33).

5. Half-angle formulas

The *Descriptio* contains two theorems that allow the logarithmic solution of the case of three given sides of an *arbitrary* spherical triangle *ABC*. One of them can be enunciated as

$$\sin^2(\alpha/2) = \frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}, \tag{12}$$

where $s = (a + b + c)/2$. In accordance with Napier's terminology we consider the side a as the "basis", α as the "angle at the top" and b and c as the "legs" of the triangle (see Figure 1). Then his version of (12) reads — in our translation (see [8], *Descriptio*, p. 47) — as follows:

If one subtracts the sum of the logarithms of the legs from the sum of the logarithms of the aggregate and the difference of half of the basis and half of the difference of the legs, then what remains is twice the logarithm of half of the angle at the top.

Here we have to observe that "logarithm" stands for logsin (see Section 3). Therefore, we must compute logsin b + logsin c and subtract this from the sum of logsin($a/2 + (b - c)/2$) and logsin($a/2 - (b - c)/2$) — the last-mentioned two sines have the "aggregate" and the difference of half of the basis and half of the difference of the legs as argument. This yields 2logsin($\alpha/2$) then. Of course, this procedure can easily be translated to formula (12).

Napier's *proof* of this formula is very short. From Regiomontanus' cosine rule (6) (where we have chosen, in contrast with Regiomontanus and Napier, the radius $R = 1$), he deduces the proportion

$$\sin b \sin c : 1 = (\text{sinvers}\ a - \text{sinvers}\ (b - c)) : \text{sinvers}\ \alpha.$$

The right hand side, however, behaves like

$$\sin(a/2 + (b - c)/2) \sin(a/2 - (b - c)/2) : \sin^2(\alpha/2),$$

as he says. Whence the assertion follows. This rather scanty argument we may extend, on the one hand, by

$$\text{sinvers}\ a - \text{sinvers}(b-c) = \cos(b-c) - \cos(a) = 2 \sin(a/2+(b-c)/2) \sin(a/2-(b-c)/2)$$

an identity known at Napier's time from prosthaphaeresis (see Section 2). On the other hand, one also knew $\text{sinvers}\ \alpha = 2 \sin^2(\alpha/2)$,

which completes the proof of Napier's assertion. In the *Descriptio* we also find the analogue of (12) for the cosine, namely,

$$\cos^2(\alpha/2) = \frac{\sin s \sin(s-a)}{\sin b \sin c}. \tag{13}$$

From a computational point of view, both formulas are equivalent, since α is uniquely determined both by its sine and its cosine, because of $0^\circ < \alpha/2 < 90^\circ$. This may be a reason why Napier abstains from a proof of (13); he says only "quod alterius loci est demonstrare" (see [8], *Descriptio*, p. 48). Napier's theorems (12) and (13) immediately give the formula

$$\tan^2(\alpha/2) = \frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)} . \tag{14}$$

In modern textbooks about spherical trigonometry the formulas (12) to (14) are called "half-angle formulas" (see, for instance, [4] p. 188, [14], p. 20). Napier did not note the last of these half-angle formulas.

6. Polar formulas

The *polar triangle* is an important concept of spherical trigonometry. It is given by the polars of the corner points of the triangle *ABC*. In order to see this, we extend the sides *c* and *b* both to a total of 90°, with end points *X*, *Y* (see Figure 4). Then the great circle through *X* and *Y* is the polar of *A*. In the same way we obtain the polars of *B* and *C*.

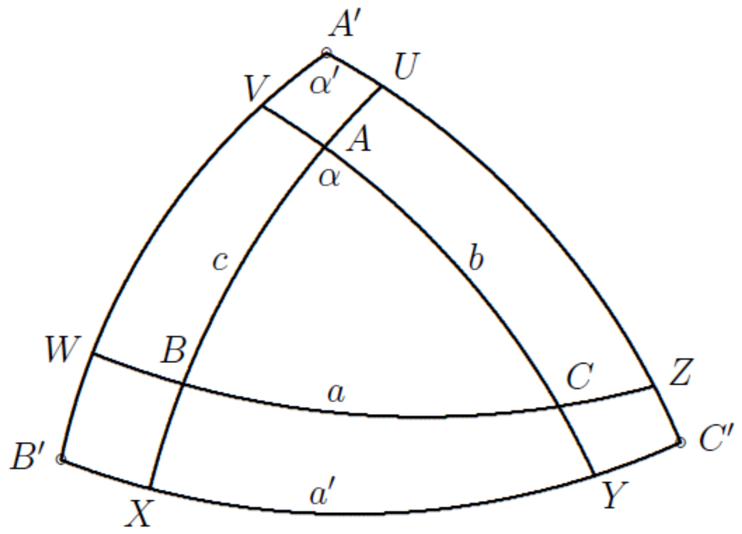

Figure 4

The polar arcs arising in this way have A', B' and C' as intersection points. By considerations analogous to those of Section 4, we find that the angle α' equals $180^\circ - a$. To this end observe that a is part of the polar of A', since the angles at Z and W are right ones. Because of $CW = 90^\circ$ and $BZ = 90^\circ$ we have $WZ = 180^\circ - a$; and, by Section 1, α' equals WZ. In a similar way we show $a' = B'C' = 180^\circ - \alpha$. In other words, the angles of the polar triangle are the supplements of the sides of the initial triangle, whereas the sides are the supplements of the angles.

The cosine rule for sides, when applied to the polar triangle, turns into the cosine rule for angles, and conversely. Therefore, it suffices to prove *one* of these rules; the other — being the polar version thereof — follows automatically. The polar versions of the half-angle formulas are called *half-side formulas*; for instance, (14) turns into the half-side formula

$$\tan^2(a/2) = -\frac{\cos\sigma\ \cos(\sigma-\alpha)}{\cos(\sigma-\beta)\cos(\sigma-\gamma)}, \tag{15}$$

where $\sigma = (\alpha + \beta + \gamma)/2$ is half of the sum of the angles. Applying the logarithm to both sides of (15) allows the logarithmic treatment of the case of three given angles. This treatment follows the pattern of the case of three given sides in the foregoing section. Half-side formulas do not occur in Napier's work, although the polar triangle was known at his time (see [3], vol. 1, pp. 182, 245). In fact, he uses polar concepts with the aim of transforming sides to angles and angles to sides, see [8], *Descriptio*, p. 55. Soon after Napier's death Henry Gellibrand solved the case of three given angles by the application of (12) to the polar triangle (see [3], vol. 2, p. 29).

7. Napier's analogies

One of Napier's most remarkable achievements is contained in [8], *Constructio*, p. 56, but without proof. It consists of two (of a total of four) so called "Napier's analogies".

The first of these analogies reads, in modern terms,

$$\tan(c/2) = \tan((a+b)/2)\cdot\frac{\cos((\alpha+\beta)/2)}{\cos((\alpha-\beta)/2)}. \tag{16}$$

Instead of the fraction on the right hand side of (16), Napier has the more complicated expression

$$\frac{\sin((\alpha-\beta)/2)\sin(\alpha+\beta)}{\sin((\alpha+\beta)/2)\sin(\alpha-\beta)},$$

which, however, amounts to the same if one applies the well known formula for the sine of a doubled angle to $(\alpha + \beta)/2$ and $(\alpha - \beta)/2$. Of course, Napier enunciates his result without formulas.

There is no difficulty in applying (16) to the case of two given sides and an angle opposed to one of these sides, and the case of two given angles and a side opposed to one of these angles. So far we have been able to solve these cases only in a scarcely elegant way by means of a quadratic equation (see Section 1). For this purpose we need the spherical *sine rule*

$$\frac{\sin a}{\sin\alpha} = \frac{\sin b}{\sin\beta} = \frac{\sin c}{\sin\gamma}, \tag{17}$$

which was known to Arabic astronomers around the year 1000. We can easily deduce this rule from Napier's rules (see Section 4). To this end we subdivide the triangle *ABC* into two right triangles with the common leg *h*, the "height" of *ABC*.

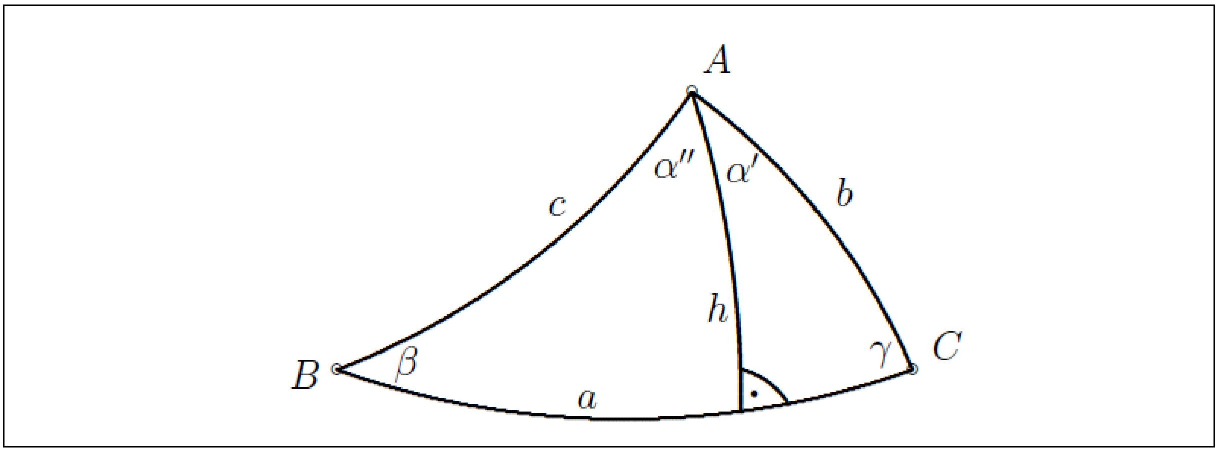

Figure 5

Indeed, Napier's rule (b) gives, for the common circular part $\bar{h} = 90^\circ - h$ of the right triangles, $\sin h = \sin c \sin \beta$ and $\sin h = \sin b \sin \gamma$, whence (17) follows by elimination of $\sin h$.

Suppose now that a, b and α are given. Then the angle β can easily be computed from (17). One has to observe that β is, in general, not uniquely determined, since the supplement of β has the same sine. This ambiguity, however, is not a disadvantage, since (16) yields a solution c for each of these two values of β, which may correspond to the geometric reality. On the other hand, if a, α, and β are given, we determine the two possible values of b and apply (16) again.

Napier himself does not mention this application of his rule, which is logarithmic throughout. In fact, he restricts himself to the logarithmic treatment of the case of two angles and the side adjacent to these. If c, α und β are given, the identity (16) yields only $a + b$. Napier's second analogy

$$\tan(c/2) = \tan((a-b)/2) \cdot \frac{\sin((\alpha+\beta))/2)}{\sin((\alpha-\beta)/2)}, \tag{18}$$

however, gives $a-b$, so both analogies together exhibit a and b. It is strange to say that the *Constructio* contains the second analogy basically in the form (18), whereas Napier did not see that his first analogy has an equally simple shape. Henry Briggs (see Section 3) wrote explanatory notes for the *Constructio*. He not only gave the simpler form (16) of the first analogy, but also the polar formulas for both analogies ([8], *Constructio*, p. 61). Thereby, the logarithmic treatment of the case of two given sides and the angle included by them becomes possible. But Briggs also did not give proofs. The first, though cumbersome, geometric proof can be found in Oughtred's *Trigonometria* of 1657 (see [3], vol. 2, p. 42). Nowadays these analogies are usually proved in an analytic way that goes back to Euler (see, for instance, [14]). A relatively simple geometric proof for the polar formula of the first analogy can be found in [2], p. 30 f.

8. Napier's heritage

Were Napier's efforts worthwhile? Indeed, fifty years ago spherical trigonometry was a matter of logarithmic computations. Typically, one used *a half-angle formula for the case of three given sides, a half-side formula for the case of three given angles, both analogies and their polar formulas for the cases of two given sides and the angle included by them or two given angles and the side adjacent to them, and, finally, one of the analogies and the sine rule in the remaining cases,* see, e.g., [14], p. 25 ff., [12], p. 43, [13], p. 269 ff. Hence we may say: Up to recent times, applied spherical trigonometry consisted of theorems of Napier and their polar formulas (if we discard the sine rule). Over the centuries other aids have been found which render the same as Napier's tools. But they do not really make the logarithmic computation simpler.

In the age of computers most of the above cases can be solved by means of the two cosine rules. But in the case of two given sides and an angle opposed to one of them the analogy (16), combined with the sine rule, is still of value. This also holds for the case of two given angles and a side opposed to one of them. Other possible solutions hardly yield simpler formulas. For instance, if b, c, β, γ are given, the cosine rules (1) and (3) give a system of linear equations for $\cos a$ and $\cos\alpha$, which ends in the formula

$$\cos a = \frac{\cos\beta\cos\gamma - \cot b\cot c}{\sin\beta\sin\gamma - \csc b\csc c}; \qquad (19)$$

here the cosecant is the reciprocal of the sine, i.e., csc = 1/sin. Although it is not difficult to keep (19) in mind, it is certainly more complicated than each of Napier's analogies.

One should also mention that a number of problems in spherical astronomy lead to *right* spherical triangles, see [13], chap. 19. In this setting, Napier's rules of Section 4 are still of value.

9. A flaw in the *Constructio*

The headline of the trigonometric section of the *Constructio* announces propositions for the solution of spherical triangles with wonderful ease (*mira facilitate*). In particular, the subdivision of triangles into right ones shall become avoidable (*triangulum sphaericum resolvere absque eiusdem divisione in duo quadrantalia aut rectangula*, see [8], *Constructio*, p. 50). This promise, however, is fulfilled on the first three pages of the said section only in a *formal* manner. In fact, the author subdivides the triangle in this way and applies his rules (of Section 4) — but without saying that the auxiliary triangles are right ones. Here the said rules are not disposable in their systematic form, i.e., Napier does not work with circular parts but, essentially, with all ten formulas for right triangles. These circumstances make it plausible that the said text dates from an early period of Napier's research. Later half-angle theorems and Napier's analogies make subdivisions of this kind superfluous.

We discuss the last two examples of the first part of the said section (Examples 11 and 12, p. 52): In Example 11, the side b and the angles α and γ of the triangle ABC are given, whereas c shall be found. As in Figure 5, the triangle is subdivided into two right triangles with the common leg h. Napier computes the angle α' between h and b by means of cos b = cot γ cot α' (which is Napier's rule (b)). Accordingly, he obtains the angle $\alpha'' = \alpha - \alpha'$ between h and c. Now the same rule gives cos α' = cot h cot b and cos α'' = cot h cot c. If one eliminates cot h in these equations, the relation tan = 1/cot yields the result, namely, tan c = cos α' tan b/cos α''.

Napier also considers the case of h falling outside of the triangle, which means $\alpha'' = \alpha+\alpha'$. He enunciates his solution as a sort of recipe, in particular, without proof. It seems plausible, however, that the many cases of right triangles that occur in this example and the ten foregoing ones were a motive for Napier to develop his rules.

In the subsequent Example 12 the setting is the same, so b, α and γ are given again. But now the angle β shall be found. Once Example 11 is solved, β could be computed by means of the sine rule (17), which, of course, was known to Napier (his Examples 3 and 8 consist in applications of this rule). Napier was certainly aware of the fact that Example 12 is no more a great challenge. Surprisingly, his solution reads, in our terminology,

$$\cos\beta = \cos b/\cot\gamma. \tag{20}$$

This is wrong, of course, since otherwise two determiners (namely, b and γ) would define the triangle, more precisely, the case of two given angles and a side opposed to one of them. In particular, α would be superfluous. The following is imaginable: Napier's solutions of Examples 11 and 12 agree in the first two lines of the text. In Example 11, these two lines lead to the correct *intermediate* result

$$\cot\alpha' = \cos b/\cot\gamma. \tag{21}$$

In Example 12, however, the false *final* result (20) turns up immediately. This suggests that Napier's original solution of Example 12 had about the same length as that of Example 11; more precisely, that three or four lines between line 2 and line 3 (the line of (20)) have disappeared. The omitted lines may have contained the following instructions:

Compute $\alpha'' = \alpha - \alpha'$, *then h by the rule* cos α' = cot b tan h, *and, finally,* cos β = cos h sin α''.

Lines number 3 and 4 (*et proveniet sinus complementi anguli B, et inde ipse angulus B quaesitus.*) may follow in their present form ([8], *Constructio*, p. 52).

The beginnings of Examples 11 and 12 being identical, a lapse of the type setter seems to us more plausible than such a blunder of Napier (who was already dead when the *Constructio* was published). This lapse would have happened with the line break of *proveniet*, a word that may have been repeated at the end of the example — just as in Example 11. Possibly this lapse escaped the corrector's attention because both Examples 11 and 12 have the same length in their present form.

One may ask why Henry Briggs, whose valuable notes are enclosed, did not see this. Possibly, he did not read this part of Napier's work thoroughly since it seemed to be of minor importance. In fact, his *trigonometric* comments all refer only to the subsequent section. Possibly, also, he wrote his commentary on the basis of a correct manuscript of Napier — which is no more existent. But these are mere conjectures.

This spoiled text was adopted, without any comment, by the English translation [9] of the *Constructio* from the 19th century. Since this translation is based on the *editio princeps* of 1619, it is clear that this flaw cannot be a consequence of the type setting of the edition [8] of 1620 which we are using.

The translation [9] also contains a helpful catalogue of the printed editions of Napier's work.

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