

Stephan Weiss & Klaus Kühn
1614 - 2014: 400 Years of Logarithms,
John Napier and his Work – a Dialogue

Chronicler (KK)

In this presentation “Calculating like in the olden days” it is our intention to put life back into arithmetic; pocket calculators and computers take away the thinking - the brain work.

In addition to collecting calculation devices, Stephan Weiss and Klaus Kühn are also the authors of several articles about these devices.

Both of us offer our presentations on a selection of living, “hands on” topics once every four months.

As a participant, you should learn about the backgrounds of the people behind the calculation devices, about the culture in those times, history and its significance today.

In today’s journey into the past we reach the turning point between the

ari uidetur in progressionē numerorum naturalī, dum seruit
progressioni.

Sed ostendenda est ista speculatio per exemplum.

-3	-2	-1	0	1	2	3	4	5	6
$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16	32	64

Posset hic fere nouus liber integer scribi de mirabilibus nu-
merorum, sed oportet ut me hic subducā, & clausis oculis abeā.
Repetam uero unum ex superioribus, ne frustra dicar fuisse in
campo isto. Sed sententia inuersa repetam quod mihi repeten-
dum uidetur.

Middle Ages and modern times. Let us now give you, in a few short words, an impression of events about 600 years ago:

In 1450 Johannes Gensfleisch, known as Gutenberg (born around 1400 - died in 1468), invented the printing press with movable metal type pieces. In 1492 Christopher Columbus (about 1451 - 1506) discovered America, marking the start of maritime exploration of the continents using star navigation. This led to more intensive worldwide trade.

In 1517 Martin Luther (an Augustinian monk from Erfurt, born 1483, died 1546) started the Reformation by posting the 95 theses on a church door in Wittenberg.

In 1518 Adam Ries (1492/3 - 1559) published *Das Rechnen auf den Linien* (Calculating on the Lines) in German (and not in Latin) so that more people can take advantage of it and he taught as a mathematician – Rechenmeister - in Erfurt.

In 1543 Nikolaus Kopernikus (1473 - 1543) published *De revolutionibus orbium coelestium*, in which he described how the earth revolves around the sun. Georg Joachim Rheticus from Feldkirch (1514 - 1574) played an important role in this publication.

2014 is Rheticus' 500th birth day.

In 1544 Michael Stifel (1487 - 1567; an Augustinian monk from Esslingen to begin with) published *Arithmetica Integra*, in which he worked out the principle of logarithms.

(Detail from Michael Stifel, *Arithmetica Integra*, Nuernberg 1544, fol. 249v)

What was going on in England and Scotland at this time?

Balance of power constantly changing between Catholics - supported by Spain -, Protestants and Anglicans. Heavy battles took place between England and Spain, which resulted in the destruction of the Spanish Armada by the English fleet in 1588.

During this time John Napier, Laird of Merchiston, was born. We are lucky enough to be able to ask him personally about his life.

Napier (SW)

(Napier steps onto the stage)

Ladies and Gentlemen, glad to be here and many thanks for the invitation. There is nothing special to say about my life. I was born in 1550 in Merchiston Tower in Edinburgh, Scotland. My father was Sir Archibald Napier of Merchiston, my mother's name was Janet Bothwell. Both were very young when I was born - just like young people are. I had a brother and a sister.

Aged 13, I was late in beginning my studies at St. Andrew's University and lived in St. Salvator's College, which is where I started taking an interest in theology. However, the other subjects taught could not satisfy my thirst for knowledge, so that I left that school and started travelling incognito in Europe and getting deeper into my university studies.

I modestly admit that I have knowledge of theology and mathematics and I am fluent in Greek and Latin.

At the age of 21 I returned to Scotland, got married and mainly lived in Gartness – Stirlingshire (50 miles west from Edinburgh) for more than 30 years. After my father's death in 1608, God bless him, I inherited Merchiston Castle as well as a substantial fortune.

Chronicler



(He interrupts)

By the way - Merchiston Tower of the Castle still stands to this day and

I never moved away from the castle again, I am not fond of travelling. I live there as laird or lord of the manor.

(slow movement with his hands)

Please excuse me, it's gout which is giving me a hard time.

I have always been interested in theology, especially in times of religious controversy and particularly John's Book of Revelation.

Roman Catholics and Reformists have been fighting each other in England and Scotland for about one hundred years. People are afraid that the Spanish Armada will attack us again and that the Spaniards will conquer England and Scotland.

I have also designed weapons to fend off attacks. Furthermore, nobody knows which direction wavering King James VI is moving in with regard to theology.

Let us move on now to John's Book of Revelation.

That is the last part of the Bible. A work of divine revelation thoroughly embodied by symbols such as

- * the four apocalyptic horsemen,
- * the book with the seven seals and
- * the whore of Babylon with the seven-headed beast.

If God gives us an encoded revelation, we have to decode it and make it easy to understand.

I am a convinced reformist, so it was all the easier for me to keep a safe distance in the interpretation of the original wording. Therefore I succeeded in making my own interpretation. I cannot go into detail here.

My transcript

A Plaine Discovery, of the whole Revelation of Saint John: set down in two treatises ... was first published in 1593.

That is the only work that I have written in my mother tongue so that

forms the centre of Edinburgh Napier University, founded in 1964.

An insertion:

The library of the Napiers was destroyed by fire at the end of 17th century, so we have no direct indication of the literature he used.

Napier 400 Memorial, presented in Gröbenzell (Germany)

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everybody can read it. Otherwise I prefer Latin because that language is more precise from a scholarly point of view and all scholars can read it.

What has been revealed to us? Here are just two essential points:

* the Last Judgment will descend on us in 1688 or 1700.

* Christ is the King of Kings and the Ruler of Rulers.

(gets himself into a rage)

* The seven-headed beast - that is Rome, the Roman Catholic Church, that is where the evil comes from...

(He interrupts strongly)

I would like to finish on that note. We do not want to expand on these circumstances now. That is your interpretation. But, **Laird of Merchiston**, the work you have done in the field of mathematics is invaluable. Let us talk about it.

If I could find time outside my theology studies I devoted it to mathematics. Multiplication and division of multi-digit numbers is difficult for astronomers because they are based on trigonometry and spherical trigonometry in particular. I explained this situation in the foreword to my *Mirifici Logarithmorum Canonis Descriptio*.

They currently use the so-called prosthaphaeretic method. This is based on trigonometric relationships and simplifies multiplications by turning them into additions. Here is an example, which you surely know:

$$\sin a * \sin b = \frac{1}{2} * [\cos(a - b) - \cos(a + b)]$$

The two product factors are normalized quantities in the region of 0 and 1, the corresponding angles a and b are sought, added and subtracted. Then one looks for the value of their trigonometric function, adds or subtracts, divides by 2 and achieves the result sought, which then has to be normalized back. Not exactly easy.

Nevertheless, astronomers are glad to have such a tool, which is why we also need highly precise trigonometric function tables such as Georg Joachim Rheticus's *Opus Palatinum*.

I thought to myself, there must be an easier way. After all, the special properties in the comparison of an arithmetic with a geometric sequence of numbers have been known for a long time. Gemma Frisius, Simon Jacob and especially Michael Stifel, who highlights these properties, have written about it.

Let me show you:

In the geometric progression each term after the first is found by multiplying the previous one by a constant. Those are the numeri.

In the arithmetic progression each term after the first is found by adding a constant to the previous one. **Those are the logarithms of the numeri.**

For 2 pairs of numbers (a,b) and (c,d) the following applies:
 if $a/b = c/d$, then $\log a - \log b = \log c - \log d$ as well (the same distances between a and b as well as between c and d)
 or if $a*d = b*c$ then also $c = (a*d)/b$.

A sample calculation for multiplication:

In our approach to the calculation we always have to go back to the full definition, that means

$a*d = c*1$ where the following values apply:

$a = 4; d = 8; b = 1; c = ?$

$\log c = \log a + \log d - \log 1 = 6 + 7 - 4 = 9; \log c = 9$ and therefore $c = 32$

To the so-called logarithms I have to add: the word originates from me. I like inventing new names and have called them proportional numbers. These logarithm numbers are defined by proportions. They represent artificial numbers, numeri artificiales in Latin, invented in order to simplify multiplications or divisions. They have no other significance!

The table above is on the one hand useful, on the other the distances between the entries are too big to be able to use them on a universal basis. The table has a structural, but not a principle problem.

					b	a	d		c		
num	geom	1/8	1/4	1/2	1	2	4	8	16	32	64
log	arithm	1	2	3	4	5	6	7	8	9	10

$$a/b = c/d \rightarrow \log a - \log b = \log c - \log d$$

Napier 400 Memorial, presented in Gröbenzell (Germany)

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I have thought about to construct such a comparison with sufficient fine graduation. In about 1594 I developed the principles of my logarithms.

In preparing the table I have the movement of two points to start with. One moves at a constant speed and represents the logarithms in certain intervals.

The other point moves at a geometrically proportional decelerating speed and represents the numeri at the same periodic intervals.

That is to say that I use the principle of steady motion. You could have calculated the geometric sequence by continued multiplication using a number less than 1. But I didn't do that. The calculation work would have been too extensive for high precision.

Then I had to calculate for a long time, set up support tables, think about precision and so on. It must have been about 20 years. And finally the table was finished. The angles 0° to 90° represent the entries into the table in 1 minute intervals, in complementary order, showing the values for sine and cosine as well their logarithms and, using the differences in the logarithms, the logarithm of the tangent as well.

(He interrupts)

Where did you get the sine values from? Or did you indeed work them out yourself?

The sine values are taken from Finke's 1583 tables and from Lansberg 1591. Sometimes I compared them in order to avoid mistakes.

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The logarithm work was published in 1614, in other words not so long ago under the title *Mirifici Logarithmorum Canonis descriptio*. Please note the double genitive referring to the noun in the title.

Here you can see an advance version, which I have kept.

My honourable friend Edward Wright, who died unfortunately in 1615, wrote a very good English translation: *A Description of the Admirable Table of Logarithms. Invented and published in Latin by that Honorable John Napier, Baron of Merchiston.*

(John Napier, *Mirifici logarithmorum canonis descriptio ejusque usus in utraque trigonometria etc.*, Edinburgh 1614, title page)

In order to avoid fractions I have applied the values for a circular radius of 10 million.

Log 1 is inequal to 0.

(repelling)

I know, I know what you are going to say.

Using logarithms is going to be difficult for numbers beyond the value range of the angle functions. That is, admittedly, a weakness in my system. And I myself have thought about modifying it. But: trigonometric relationships can be worked out more easily with my system. I have therefore adapted a few trigonometric relationships to the logarithms.

And as log *sinus totus*, i.e. sine 90° equals 0, some calculations become more simple with logarithms containing the sinus totus.

As far as advantages and disadvantages are concerned, ultimately what I have written in the descriptio counts: *Nihil in ortu perfectum* (nothing is perfect to begin with).

The explanations of my calculations entitled *Mirifici logarithmorum canonis constructio* have already been completed and will be published shortly.



I also have to add that I agreed substantial amendments to a future logarithm table when, in 1615, I met my honourable friend and colleague, Henry Briggs, Professor of Geometry at Gresham College in London, which was only a few years old – maybe you know him .

And this is how it happened ...

Honourable Mr. Briggs, I have to say you are right.

I know my choice was far too geared up to being used in trigonometry. I also thought about making such an amendment. Your proposal has the advantage of using a base of 10, honourable colleague Briggs. If you want to do it that way, then I would make

$\log 1 = 0$,

$\log 10 = 1$ or $1 * \text{radius}$,

$\log 100 = 2$ or $2 * \text{radius}$ and so on.

In that case the logarithms for numbers larger than 1 are always positive and the bothersome companion $\log 1$ becomes zero and no longer appears.

I am too old to start working on such a table again.

(The chronicler dresses into a black jacket and becomes Mr. Briggs.
He stands up)

After reading John Napier's *Descriptio*, I went to visit him because I was so thrilled about the idea – with one exception. That is why I said to him: Sir John, when using logarithms you put the focus on trigonometric relationships. If however you are calculating beyond the value range of sine 0 to 1 it becomes difficult. Furthermore, above 1 or the radius of 10 million the logarithms are negative and therefore harder to handle.

I thought about making the following amendment. One makes

$\log 1 = 10$ or $10 * \text{radius}$,

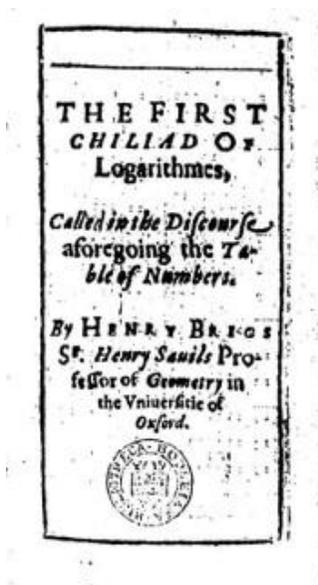
$\log 10 = 9$ or $9 * \text{Radius}$ and so on up to

$\log (10 \text{ billion}) = 0$.

That makes calculating in steps of 10 easier.

If I think about it, your proposal is better than mine, so I will include it in my new table.

(The chronicler strips off his jacket and sits down)



Due to ill health John Napier was no longer in a position to re-calculate everything. That is why Briggs started to calculate the table all over again according to Napier's proposal in a classification which we still use today.

(Henry Briggs, *Logarithmorum Chilas prima*, Edinburgh 1617, title page)



The following has to be said about the development and evaluation of logarithms as well as about Napier's comments:

What Napier can't know is that at the same time a Swiss gentleman called Jost Bürgi (1552 - 1632) was also calculating a logarithm table but on a different basis to Napier's. However, he didn't publish his until later, namely in 1620.

(Jost Bürgi, *Arithmetische und Geometrische Progress Tabulen, sambt gründlichem Unterricht, wie solche nützlich in allerley Rechnungen zugebrauchen und verstanden werden sol*, Prag 1620, title page)

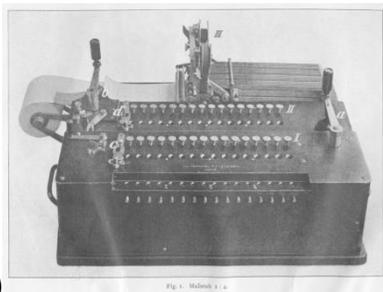
When the *Constructio*, or to give it its full title, *Mirifici Logarithmorum Canonis constructio* was published Napier was no longer alive. It was published by his son in 1619

The definition of logarithms being the proportion in a comparison of an

arithmetic with a geometric progression was maintained until the beginning of the 18th century. It was only after applying the concept of a function and using the expression of a power roundabout the first half of the 18th century that logarithms were given the definition used for them today.

Logarithms, the “artificial numbers” in former times, are no longer such things. They have become an essential part of many calculations and formulations of regulations in mathematics and natural science. Here is just a brief mention of the history of the tables of some of the more important table makers (logarithm calculators):

- * Johannes Kepler (from Germany) 1624,
- * Ezechiel de Decker and Adrian Vlacq (from Holland) 1626,
- * Abraham Sharp (from England) 1717,
- * Juri Vega (from Slovenia, part of Austria at that time ; therefore published in German and Latin 1783,
- * Jean Peters (from Germany) 1910,
- * Arnold Noah Lowan (from USA) (log nat) 1941.



From the 19th century onwards logarithms were calculated with mechanical aid, such as difference engines or computers for example.

(Left Photo from Julius Bauschinger, Jean Peters, (ed.), *Logarithmisch-Trigonometrische Tafeln mit acht Dezimalstellen*, 2 vols, Leipzig 1910)

(Right Photo from Alexander John Thompson, *Logarithmetica Britannica, being a standard table of logarithms to twenty decimal places*, Vol. 1: numbers 10.000 to 50.000, vol. 2: numbers 50.000 to 100.000. Cambridge University Press 1952)

You didn't have such assistance at your disposal. Sir John, you had no way of making calculation simple.

No, I didn't, which is why I made numbering rods to help me calculate ...

Working out logarithms was tough going as you can probably imagine. Doing it all by hand took decades; as far as I can remember about 20 years.

I therefore designed a simple calculation tool called *Rabdologia*, which is an artificial word taken from Greek *rabdos* = rod and *logia* = knowledge of, information about.

I took the idea about the rods from the method of two-dimensional lattice multiplication, which you surely know. One draws a rectangle, inserts the product factors at the top and at the edges on the right and the partial products of the digits in the lattice. Then you just add diagonally. The lattice only has to be cut into strips and restrict oneself to a single-digit factor. I myself thought nothing of my invention, but when I showed it to friends they urged me to publish my information on these rods in any event. I therefore wrote a little book about *Rabdologia*, which appeared in 1617 with an explicative title: *Rabdologiae seu numerationis per virgulas libri duo*. (He shows the book).

(John Napier, *Rabdologiae seu numeratio per virgulas libri duo*, Edinburgh 1617, title page)

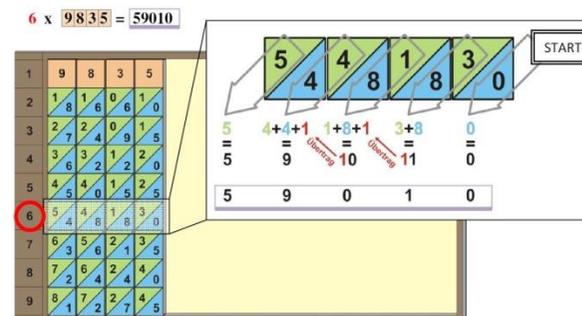


The whole system consists of a few rods with a small multiplication table. On one side of the rod the 2 to 9 multiples of the number at the top are listed underneath each other. If you lay the rods side by side so that the numbers at the top show a multi-digit figure, then you can read the 2 to 9 times multiplication of this figure in the rows further down. You only have to add the diagonal numbers to the amount carried over. (He shows a set of rods).

It is important to realize that you need not necessarily know the small multiplication table, you just need to know how to add. Division is not difficult either because you can combine the multiples of the given quotients and compare them with the dividend. Two more rods are helpful in calculating square and cubic roots. For practical purposes the rods are either made of wood or bone or ivory. Precision work is not necessary. That is why some people call them *Napier's bones*. I don't really know what to make of that. Somebody found it funny.

If you want to extend the principle of the calculating rods to multi-digit

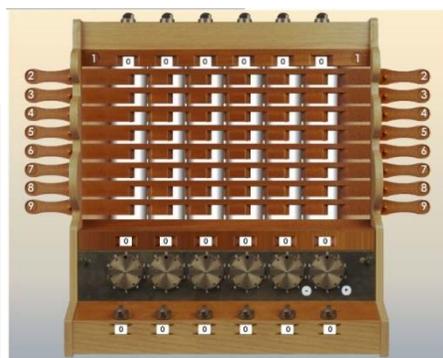
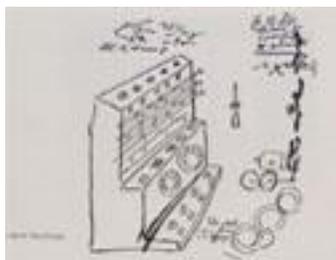
Multiplikation (einstellige Zahl):



product factors, you arrive at a system that I have called *Promptuarium Multiplicationis*. Promptuarium is -- not a name I invented, it already exists and means as much as store, immediately accessible. The rods are horizontal and vertical. If you lay these rods under each other, you arrive at the picture of a complete lattice multiplication.

In *Arithmetica localis* I illustrate how to display numbers graphically in an addition system independently of a positional notation system and how to calculate with movements in a two-dimensional system.

I cannot judge to what extent mathematicians will accept my calculation tools. I have only made them available for use.



Here is a brief description of the way forward and the valuation of the calculation tools:

Napier's numbering rods spread quickly. After only one year the German version of *Künstlichen Rechenstäblein* (artificial little calculation sticks) is published in 1618 described by Franz Kessler (1580 - 1650). Further editions follow in other languages.

As early as 1624 Wilhelm Schickard outlined a calculating machine with a tens carry mechanism on the basis of the Napier numbering rods.

(Left: Original drawing taken from Friedrich Seck (ed.) *Wilhelm Schickard 1592-1635, Astronom, Geograph, Orientalist, Erfinder der Rechenmaschine*, Tuebingen 1978)

(Right: A later reproduction of Schickard's Calculating Machine)

Napier's numbering rods become highly well known and are always mentioned in every arithmetic textbook in subsequent times.

From the end of the 19th and the beginning of the 20th century this idea is again included in numerous patents for calculation aids. Their acquisition costs were namely much lower than those of calculating machines.

The *Promptuarium*, on the other hand, remained unknown to a large extent.

(he reflects a bit)

What else have I done in the field of mathematics?...

I started working on an *ars logistica*, including the methodology of arithmetic and algebra. But then I considered the work on logarithms to be more important and put that other work off. I don't yet know when I will publish it.

On the subject of decimal fractions:

the illustration of decimal fractions has been inconsistent up until now and has therefore caused problems. A marking of decimal places is used with hexadecimal numbers, for example $7^{\circ}22''$. Simon Stevin applied this method to decimal fractions using decimal place marking, but this is laborious. I prefer to use a decimal point like others do. The places result from the sequence of numbers and do not have to be marked.

In trigonometry I have studied spherical triangles. They are important for navigators and astronomers. My aim was to reorganize the relationships so that they could be adapted to the use logarithms. Furthermore, I managed to draft new relationships with spherical triangles.

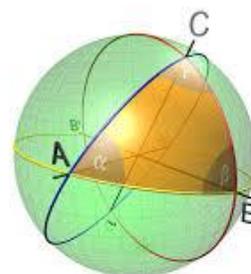
(spherical triangle; wikipedia)

It is much too big and hard to handle.

The *Arithmetica localis* is a computational plaything and is no longer mentioned in subsequent literature.

Besides his calculating aids Napier's influence on mathematics should be mentioned too.

Sir John, you have given mathematics other new ideas as well.



Ars logistica wasn't published until centuries later which - despite being modern - prevented it from having any influence on contemporary mathematics.

Now it is time for me to evaluate Napier's work for the following centuries. With logarithms and the numbering rods John Napier has had a significant influence on calculation techniques of the following centuries. That applies especially but not exclusively to calculations in astronomy, cartography and navigation.

Logarithms provide the basis for the development of the slide rules used for the following 350 years and logarithm tables of which over 3,000 are documented in *Collectanea de Logarithmis*.

Logarithms are an essential part of many laws of nature.

Logarithms have been worked out in many different ways and have added enormous value to the mathematical way of thinking. Logarithms are still in use today, for example

- * in determining pH levels in chemistry,
- * in calculating the strength of earthquakes on the Richter scale,
- * in measuring radioactive decay,
- * in determining volume levels in decibels,
- * to demonstrate the relationship between stimulus and perception (Weber-Fechner law),
- * for interest and compound interest calculations.

Logarithms also play an important role with music and acoustics as well as in high performance computing.

One question arises when looking at this extensive and important mathematical work of one single man, which he should answer himself.
(He stands up)

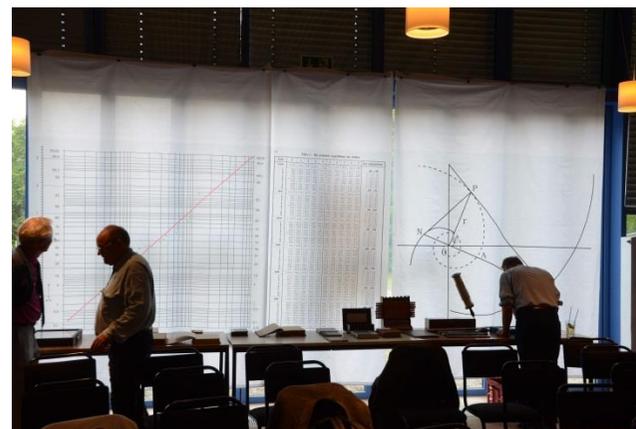
Sir John, what was your biggest work?

(He stands up) My biggest work?

Without a doubt, my biggest work was my interpretation of John's Book of Revelation.



Audience



Exhibition

Many thanks for coming and for your attention.



Chronicler (left) and Napier (right)

References:

- (1) Hamann, Christel: http://www.rechnerlexikon.de/artikel/Differenzenmaschine_Hamann
- (2) Thompson, A.J.: http://www.rechnerlexikon.de/artikel/Differenzenmaschine_Thompson
- (3) John Napier portrait with Courtesy The University of Edinburgh