Napier and the Rule of Three

John Napier invented his version of logarithms to facilitate calculations which involved an equality of ratios. Initially those ratios were of the sines of angles, with the calculations he had in mind solely related to the solution of spherical triangles: he was later to realise that his invention did not inherently suffer this restriction and could be brought to use with plane triangles and with calculations of ratios in general. Specifically, if three of the four letters in the ratio $\frac{a}{b} = \frac{c}{d}$ are known, the fourth is obtainable, particularly with the use of Napier's logarithms, which converted each ratio to a difference. In considering such calculations, Napier was immersed in the *Rule of Three* and he also considered generalisations of it, the Rule of Five, Seven…, where problems can become distinctly perplexing and their solutions even more so.

As well as being at the core of his motivation for the invention of logarithms, the Rule of Three appeared in Rabdolgia, with regard to the use of his rods, but we should look to Arte Logistica for his more detailed study, and to the problems that he provided:

Example 1: If a man walks 4 miles in 3 hours, how many miles will he walk in 6 hours?

Example 2: If 6 cows eat 3 measures of hay in 4 days, how many cows can be fed on 5 measures of hay in 2 days?

Example 3: 20 Scottish shillings make £1, £2 are equal to 3 marks and 5 marks are worth 1 crown. How many shillings are 9 crowns worth?

And later,

If 4 builders have constructed a wall 6 feet high, 48 spans long, in 42 days; it is sought, in how many days will 5 builders construct a wall 9 feet high, 50 spans long?

Omitting his Example 3, we have examples of, respectively, the Rule of Three, Rule of Five and Rule of Seven; including it, we have an example of one of the several associated rules; in this case, Conjoined Proportion. There was also Medial Proportion, Partitive Proportion, the Rule of Practice etc., each of which by Napier's time had a long history and each of which survived into the early 20th century as standard arithmetic techniques. For example, Problem 69 of the Rhind Papyrus of ~1650 BCE is:

With 3 half-pecks[1] *of flour 80 loafs of bread can be made. How much flour is needed for 1 loaf? How many loafs can be made from 1 half-peck of flour?*

The 12th century CE Hindu mathematical luminary, Bhaskara challenged with:

A palas and a half[2] of saffron are purchased for three sevenths of a niska.[3] How many will be purchased for nine niskas?

Passing to the 19th century, the second autobiographical note of Abraham Lincoln is dated December 20 1859 and contains the following lines:

There were some schools, so called; but no qualification was ever required of a teacher beyond "readin, writin, and cipherin" to the Rule of Three. If a straggler supposed to understand latin happened to sojourn in the neighborhood, he was looked upon as a wizzard. There was absolutely nothing to excite ambition for education. Of course when I came of age I did not know much. Still somehow, I could read, write, and cipher to the Rule of Three; but that was all.

In his own autobiography Charles Darwin commented of mathematics that *it was repugnant to me*; apt to dismiss complex mathematical arguments, he wrote to a friend *I have no faith in anything short of actual measurement and the Rule of Three*.

Napier's own approach was two-fold: a demonstration of his own general algorithm for the solution of such problems, which could replace the piecemeal treatment of previous authors, and the provision of motivating examples in the simplification of the product of fractions, using cancellation. As to his algorithm:

Rule 1: First draw a line, which is used to group and separate the data and the unknowns according to the rules that follow.

Rule 2: If there are two quantities one of which increases as the other decreases, they must be placed side by side on the same side of the line

Rule 3: Two quantities which both increase, or both decrease simultaneously, are inserted on opposite sides of the line

Rule 4: Of two quantities of the same kind, one must be above the line, and the other below the line.

For his problems, he then provided the solutions according to his algorithm:

Example 1:

Apply Rule 1 and we have $\dfrac{\text{3 hours} \;\; \text{4 miles}}{\text{6 hours} \;\; \text{how many miles?}}$

Since the hours and miles increase or decrease in proportion, Rule 3 applies and the 3 hours and 6 hours must be interchanged.

This gives $\dfrac{\text{6 hours} \;\; \text{4 miles}}{\text{3 hours how many miles?}}$

Finally, the answer is $\dfrac{6 \times 4}{3} = 8$ miles.

Example 2:

Apply Rule 1 and we have $\dfrac{\text{6 cows} \;\; \text{3 measures} \;\; \text{4 days}}{\text{How many cows?} \;\; \text{5 measures 2 days}}$

Since the number of cows and the measures of hay increase or decrease together Rule 3 applies and the 3 and 5 must be inverted. As the number of cows increase, the number of days on which they can be fed by the same amount of fodder decreases. Rule 2 applies and no change is needed.

This gives $\dfrac{\text{6 cows 5 measures 4 days}}{\text{How many cows? 3 measures 2 days}}$

Finally, the answer is $\dfrac{6\times 5\times 4}{3\times 2}=20$ cows.

Example 3:

Apply Rule 1 and we have $\dfrac{20s.\ \text{£1 3 marks 1 crown}}{\text{How many s.? £2 5 marks 9 crowns}}$

Here, a change in the number of shillings must cause an equivalent increase or decrease in the number of pounds. Consequently, the £2 and its equivalent in marks also change in value. This is also true of the 5 marks and the crown to which it is equal. Finally, the required number of shillings must also change since they must be equal to 9 crowns.

And so from each pair, one quantity must be placed below the line and the other above the line as you may see.

Rules 3 and 4 apply in this case: $\dfrac{20s.\ \text{£2 5 marks 9 crowns}}{\text{How many s.? £1 3 marks 1 crown}}$

Finally, the answer is $\dfrac{20\times 2\times 5\times 9}{1\times 3\times 1}=600$ Scottish shillings.

The reader may wish to test their mastery of Napier's approach with his final example, but we will attack it, and the others, using modern ideas.

The problems each involve a set of measurable items (days, men, length, time, ...) with one of them given a distinguished place; each of the remaining items is either directly or inversely proportional to the distinguished item (and so directly proportional to each other). Suppose that we list the measurable items as $X, X_1, X_2, X_3, ..., X_n$, with X the distinguished one. Now partition these as $X, [X_1, X_2, X_3, ..., X_k], [X_{k+1}, X_{k+2}, X_{k+3}, ..., X_n]$, where X is directly proportional to each member in the first sub-list and inversely proportional to each member in the second (with either possibly being empty). Then

$$X \propto X_1 X_2 X_3 ..., X_k \times \frac{1}{X_{k+1} X_{k+2} X_{k+3} ..., X_n} = K\frac{X_1 X_2 X_3 ..., X_k}{X_{k+1} X_{k+2} X_{k+3} ... X_n}$$

and so $\frac{XX_{k+1}X_{k+2}X_{k+3}...X_n}{X_1X_2X_3...X_k} = K$.

The data is typically supplied in two sentences: the first provides a set of known instances $x, x_1, x_2, ..., x_n$ of the items, with x distinguished; the second provides a corresponding set of instances $y, y_1, y_2, ..., y_n$ with y the unknown corresponding to the known x; we require to find y. Substituting these values into the above equation yields

$$\frac{xx_{k+1}x_{k+2}x_{k+3}..x_n}{x_1x_2x_3..x_k} = \frac{yy_{k+1}y_{k+2}y_{k+3}...y_n}{y_1y_2y_3...y_k} = (K)$$

and so

$$y = x\frac{(x_{k+1}x_{k+2}x_{k+3}..x_n)\times(y_1y_2y_3...y_k)}{(x_1x_2x_3..x_k)\times(y_{k+1}y_{k+2}y_{k+3}..y_n)}$$

And we have the Rule of $2n+1$:

$$y = x\frac{\prod_{x\in I} x \times \prod_{y\in D} y}{\prod_{x\in D} x \times \prod_{y\in I} y}$$

where D are values of items which are in direct and I those in inverse proportion to X.

Thus equipped, we will consider Napier's examples.

Example 1: If a man walks 4 miles in 3 hours, how many miles will he walk in 6 hours?

For clarity, we construct a table which summarises the information and flag whether a quantity is directly or inversely proportional to x:

X=Distance	X_1 = time
$x = 4$	3
y	6
	D

So, using the formula above, $y = 4 \times \frac{6}{3} = 8$.

Example 2: If 6 cows eat 3 measures of hay in 4 days, how many cows can be fed on 5 measures of hay in 2 days?

Here, the table is:

$X =$ no. of cows	$X_1 =$ amount of hay	$X_2 =$ no. of days
$x = 6$	3	4
y	5	2
	D	I

So, the formula leads to $y = 6 \times \frac{4 \times 5}{3 \times 2} = 20$

We move to that later example:

If 4 builders have constructed a wall 6 feet high, 48 spans long, in 42 days; it is sought, in how many days will 5 builders construct a wall 9 feet high, 50 spans long?

The table is

$X_1 =$ no. of builders	$X_2 =$ ht. of wall	$X_3 =$ lth. of wall	$X =$ no. days
4	6	48	$x = 42$
5	9	50	y
I	D	D	

So, the formula gives $y = 42 \times \frac{4 \times (9 \times 50)}{(6 \times 48) \times 5} = 52\frac{1}{2}$

And, for reinforcement, let us consider an example from a 19th century school textbook:

If 180 men in 6 days of 10 hours each can dig a trench of 200 yards long, 3 wide and 2 deep; in how many days of 8 hours long will 100 men dig a trench of 360 yards long, 4 wide and 3 deep?

The table is

$X =$ no. of days	$X_1 =$ no. of men	$X_2 =$ no. of hours	$X_3 =$ length	$X_4 =$ width	$X_5 =$ height
$x = 6$	180	10	200	3	2
y	100	8	360	4	3
	I	I	D	D	D

And so, the formula gives $y = 6 \times \frac{(180 \times 10) \times (360 \times 4 \times 3)}{(200 \times 3 \times 2) \times (100 \times 8)} = 48\frac{3}{5}$.

Finally we look to Napier's Example 3 of Conjoined Proportion, which is somewhat subtle.

We tabulate the information to arrive at

No. Shillings	No. £	No. Marks	No. Crowns
$x = 20$	1	y_1	y_2
	2	3	
		5	1
y			9

Part of which is

No. £	No. Marks
1	y_1
2	$x_1 = 3$
D	

And so, $y_1 = 3 \times \frac{1}{2}$

A second part is

No. Marks	No. Crowns
y_1	y_2
5	$x_2 = 1$
D	

And so, $y_2 = 1 \times \frac{y_1}{5}$

And the final part is

No. Shillings	No. Crowns
$x = 20$	y_2
y	9
	D

$$y_2 = 20 \times \frac{9}{y_2} = 20 \times \frac{9}{1 \times \frac{y_1}{5}} = 20 \times \frac{9}{1 \times \frac{3 \times \frac{1}{2}}{5}} = \frac{20 \times 2 \times 5 \times 9}{1 \times 3 \times 1} = 600$$

The numerators and denominators of the fractions which arise naturally contain products of integers and with *Arte Logistica* defined by Napier as *the art of computing well* these fractions provided fertile ground for his demonstration of the sound technique of fraction cancellation: cancel first, then multiply.

The Rule of Three continues to exist, of course, in textbooks of elementary mathematics, disguised as direct and indirect proportion: the difference, though, is that we now evaluate that constant of proportionality whereas it was implicitly subsumed in days gone by. With its demise as a defined arithmetic method, the importance of the Rule of Three in the invention of logarithms is easily overlooked and, like their inventor, easily forgotten.

[Adapted from *John Napier: Life, Logarithms and Legacy*]

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[1] 1 half-peck~4.8 litres

[2] 1pala~62 g.

[3] A niska was a standardized gold coin