'More of a Lark than a Labour': Napier's Chessboard Abacus

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Introduction

What Napier called 'location arithmetic', described in the final part of his *Rabdology*, is both highly original in concept and strikingly elegant in the practical execution. To appreciate these qualities better we first look briefly at the use of chequer-boards and counting-boards. These were common-place in the centuries leading up to Napier's lifetime and adopted by accountants, tradesmen, farmers, map-makers, surveyors, navigators and a host of others requiring help with fast, reliable, arithmetical calculations.

The origin of the term 'exchequer', as is well-known, lies in the use of a chequer-board surface for calculation with counters. It is interesting in the context of introducing Napier's use of such a board to quote from a 12th century source, *The Dialogue concerning the Exchequer* (FitzNeal c.1180):
Disciple. What is the exchequer?
Master. The exchequer is a quadrangular surface about ten feet in length, five in breadth, placed before those who sit around it in the manner of a table, and all around it it has an edge about the height of one's four fingers, lest any thing placed upon it should fall off. There is placed over the top of the exchequer, moreover, a cloth bought at the Easter term, not an ordinary one but a black one marked with stripes, the stripes being distant from each other the space of a foot or the breadth of a hand. In the spaces moreover are counters placed according to their values; […]
D. What is the reason of this name?
M. No truer one occurs to me at present than that it has a shape similar to that of a chess board.
D. Would the prudence of the ancients ever have called it so for its shape alone, when it might for a similar reason be called a table (tabularium)?
M. I was right in calling thee painstaking. There is another, but a more hidden reason. For just as, in a game of chess, there are certain grades of combatants and they proceed or stand still by certain laws or limitations, […].

The Master goes on to explain that the chess allusion is appropriate because the potential 'battle' between treasurer and sheriff in resolving accounts mirrors the battle of chess opponents. As we shall see, Napier exploits a different feature of chess to justify his reference to 'location arithmetic' (as he calls it) as being performed on a chessboard.

It appears that the 'exchequer' described above was probably something specific to government accounting and relatively elaborate. Much more common and flexible was the simple counting board consisting of horizontal lines marked in wood or stone. One of the earliest examples known of such a board is the Salamis Tablet dating from 300BC (Anon Salamis Tablet). But successor artefacts were used widely in Europe in numerous contexts for money and commodities in the centuries leading up to the 17C during which the positional system of Hindu-Arabic numerals written and manipulated on paper became

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predominant. A good description of the counting board in the medieval Scottish context is given in the final sections of Goodare 1993. A generic decimal example of the counting board in use is given in that paper and reproduced here as Figure 1 below:

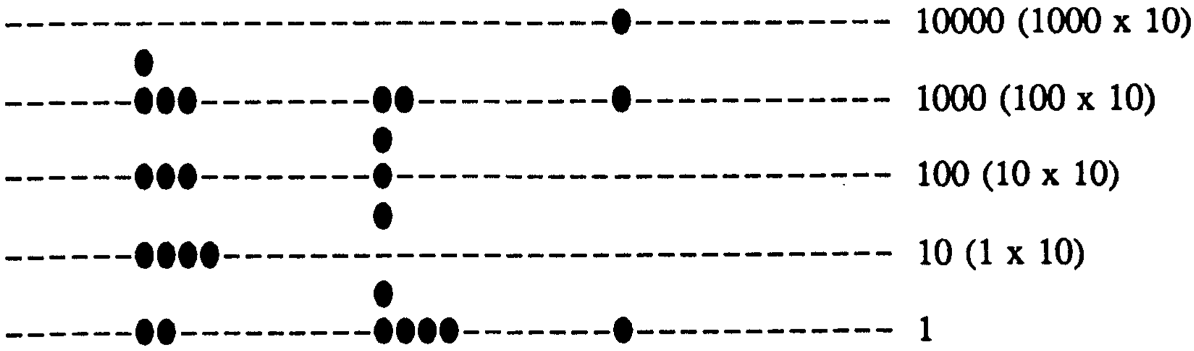

Figure 1

Successive lines hold counters that have a value that is a power of ten. Up to four counters may be placed on a line, if five are needed one counter above the line is used. The counters in Figure 1 thus represent the addition of 8342 (on the left) and 2659 (centre) to make 11,001 (on the right). This itself is an example taken from Recorde 1618 where there is a more detailed description (on p.244 ff of the downloadable edition indicated in the References) including the intermediate states. The grouping of counters may get hard to disambiguate if they get more numerous and closer together: vertical lines can help separation as Recorde indicates. The board would then become a grid. The horizontal lines can easily be adapted to successive ranks of units - for examples pounds, shillings and pence - or units of time (seconds, minutes, hours, days etc), distance, weights, volumes etc. The chequer-board form was exclusively used for accounting (Goodare 1993) where the interpretation of squares was fixed and governed by currency denominations. Clearly it could also be used like a conventional abacus for adding up a list numbers or quantities by introducing successive new numbers on the left and adding to the existing total to make a new total. Multiplication and division could be achieved by successive doubling and halving. Although the counters each represented a unit on its own line there is no suggestion of a binary interpretation. The basic gathering together in addition is crude and cumbersome. Of course, the 'advantage' was the avoidance of any knowledge of Hindu-Arabic numerals and any knowledge of multiplication tables. Goodare remarks in passing, *This is not the place to discuss John Napier of Merchiston, but logarithms could not have been invented on the counting-board.* (*ibid* p. 407) Nor, we might add, could the chessboard abacus have been invented on the counting board. It is a radically new conception.

Location numerals

We are not giving a comprehensive account here of what is in the *Location Arithmetic*, the final part of the *Rabdology,* but we seek to present the main ideas, and in doing that to be close to the original and in some respects – which will be noted – to be even more faithful to the Latin original of 1617. At the same time we are seeking, in spite of inevitable overlap, to be somewhat complementary to the treatment in Chapter 5 of Havil 2014.

The methods of the rods and the promptuary (described in the first two books of the *Rabdology*) were ingenious and also already encoded knowledge of multiplication tables. At least the rods – in various forms – were popular through the 17th century, declining in use only when purely symbolic methods, and the memorising of multiplication tables, became widespread during the 18th century.

However, when we come to the chessboard abacus, the 'location arithmetic', we are immediately aware of something radically new. Something of which he deservedly felt rather proud. We quote from his own *Preface*:
As this (the chessboard abacus) *performs all the more difficult operations of common arithmetic on a chessboard it might well be described as more of a lark than a labour […] it carries out […] multiplication, division, and (yes!) the extraction of roots purely by moving counters from place to place. There is one small difficulty […] the numbers it uses differ from ordinary numbers, so that one must begin by expressing ordinary numbers in the new form and end by reducing them to common form.*

It's likely that the phrase here 'one small difficulty' is said with tongue in cheek. The difficulty was that the 'new form' to which ordinary numerals had to be converted (and vice versa) was what he called 'location numerals'. But however familiar Napier had become with location numerals – doubtless with much practice – this difficulty was likely to be found a major obstacle for readers in the 17th century, both a conceptual and a practical obstacle.

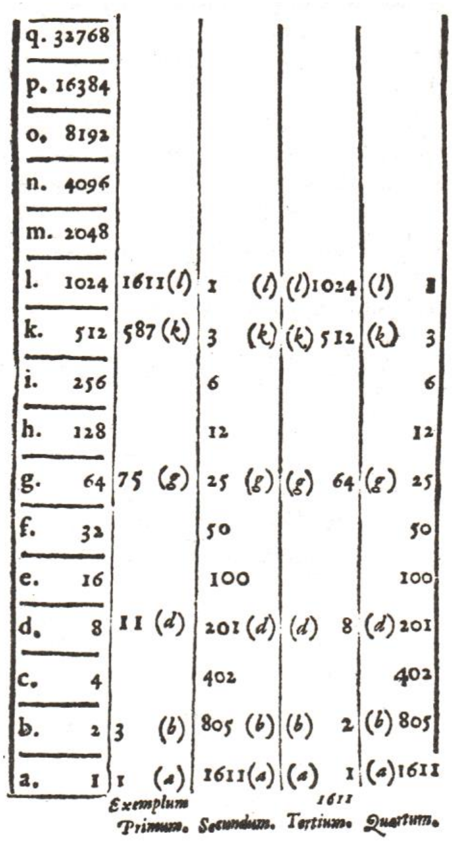

Figure 2

A rod divided into equal sized cells is introduced (Figure 2 left column) where each cell is labelled with both a power of 2 and a letter of the alphabet, so a = 1, b= 2, c= 4, d = 8, etc. Thus the decimal number 19 may be denoted by the string *abe* – the values of each letter in the string are simply added up. So location numerals are not at all the same as modern binary numerals. They afford two simultaneous interpretations: as a value derived from the values of the letters in the string, and as a disposition of counters at the locations on the rod corresponding to the letters in the string. The value of a letter in a string is fixed and does not depend on its position in the string. It is a clever system as the natural ways of extending or abbreviating a string prepare the way for what is needed to be done with counters on the rod. Thus if we wanted to add *abe* and *acd* they first up as just *f* (19+13=32). To add or subtract two numbers on such a rod we can simply place the counters for each number at the appropriate cells and manipulate them as we would with the letters in their strings. To subtract *abe* from *f* we need to extend *f* = *ee* = *dde* = *ccde* = *bbcde* = *aabcde*, then subtraction of *abe* leaves *acd*. Neither abbreviation nor extension affect the value of a string. The location numeral is a dual representation – simultaneously a string of letters and a configuration of counters on the rod.

Converting between Ordinary Numerals and Location Numerals

For the conversion of ordinary numerals to location numerals Napier describes the two methods most commonly taught to school children today: what he calls the methods 'by subtraction or by division by 2'. In describing a subtraction method he makes a mistake in saying, '[we deduct] from the given number whichever number on the rod is nearest to but less than the given number … '. He should have said '[we deduct] from the given number the largest number on the rod which is not greater than the given number'. For the method by division by 2 he describes a procedure exactly equivalent to the customary school one of writing down remainders on repeated divisions by 2 and reading the remainders in reverse. Of course, that gives a standard binary numeral which is not what Napier wants so he simply picks out the powers of 2 starting at *a* for the first remainder of 1 and jumping a letter or cell when the remainder is 0.

For the conversion the other way from location numerals to ordinary numerals the first method given is the obvious one of simply adding all the relevant powers of 2. The second method he illustrates is by performing the exact inverse of the operations used in division by 2 for conversion from ordinary numerals to location numerals. So the method is not in doubt, but it is unusual – at least for this author. We give it here in full to illustrate the detail to which Napier is willing to give to make his methods clear to his readers (see Figure 2).

Example 4 Given a number expressed in local notation as l, k, g, d, b, a, find the number by this method. For l double 1, which makes 2. To this, add 1 for the counter on k, which makes 3. Double this again which makes 6, but add nothing as position i is vacant. Double the 6, which makes 12, but again add nothing as position h is vacant. Double this, which makes 24, and add 1 for the counter found in psition g. Double 25, which makes 50, as position f is vacant. Double again, which gives 100, as position e is vacant. Double again, and add1 because of the counter on d. Doube 201, which gives 402 for the vacant position c. Double again, which makes 804, and add 1 for the counter on b. Finally double 805, which gives 1610. This must be increased by unity because of the counter on a. The number 1611, which thus arises for position a, is the required number concealed by the counters and the letters a, b, d, g, k, l.

Multiplication and Division

The board diagrams in the Latin original (Napier 1617) were taken over and reprinted in the translation by Richardson (Napier 1990) with two changes. The original vertex labels were signs of the zodiac ♈, ♉, ♊, ♋ (from left to right aries, taurus, gemini and cancer) but these were replaced in translation by A, B, C, D respectively. The originals had edges (roughly) parallel to the edges of the page, the reprinted versions were rotated 45 degrees clockwise rendering the phrases 'lefthand' and 'righthand' margins natural. For our diagrams here (Figures 3, 4, 5, 6) we have decided, for the sake of authenticity and context, to revert to the original labelling and orientation. However, we continue often to use reference to the left and right margins so the reader is advised to view the board as if from the bottom right vertex (labelled ♈ looking towards the vertex at ♊.

For working with location numerals on the board we imagine two rods set up as margins just outside the board below the edges ♈♉and ♈♋ (see Figure 3). One of these rods will hold the multiplier and the other the multiplicand. Napier has explained in a clear preparatory chapter (Chapter 7) that he is considering two kinds of movement of the counters: direct or diagonal. Direct movement is movement like the rook (or 'tower-bearing elephant' as Napier has the name in Latin) in chess which is parallel to one board edge and perpendicular to another. Diagonal movement is movement like the bishop (the 'archer') in chess. The first simple example given by Napier is to multiply 19 (*abe*) by 13 (*acd*). There is possibly an error in Napier's original diagram (a copy of which we are using here) because the value 16 on margin ♈♉is labelled *c* instead of *e*.[2] Suppose that is corrected, and imagine counters placed at each of *a, b, e* on ♈♉and *a, c, d* on ♈♋. Then imagine we drew the lines of direct motion from *a, b, e* on ♈♉, parallel to ♈♋and from *a, c, d*, on ♈♋parallel to ♈♉. These lines will have nine squares of intersection on the board. Remove the counters on the margins and place counters on the nine squares of intersection. Napier explains that every square on the board is associated with three values: the two projections onto the margins from which it is composed and the value on either margin when moving right or left off the board and parallel to ♉♋. This latter value is always the product of the two former values. For example, the intersection of the direct movement of *e* (16) on ♈♉and that of *c* (4) on ♈♋is a square that moves off the board diagonally, on either margin, to g (64). It is highly instructive, and essential, that the reader proves (by drawing suitable diagrams or otherwise) the general result for herself.

Now it should be clear that the whole product of the two numbers on the margins is represented by the nine counters in this rectangular configuration. It may be equally seen as 13 copies of 19 or 19 copies of 13. Napier calls any row or column of the rectangle of counters on the board a *segment* of the rectangle.

The final product can be totalled conveniently by moving all nine counters off diagonally to one of the margins – let us say the right-hand one. When two or more counters accumulate at the same cell on the right margin every pair of counters are replaced by one at the next higher cell. The result 247 is shown in powers of 2 in Figure 3.

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[2]In fact the symbol is blurred in the original diagram, it is 're-written' in the 1990 translation but erroneously as a 'c'.

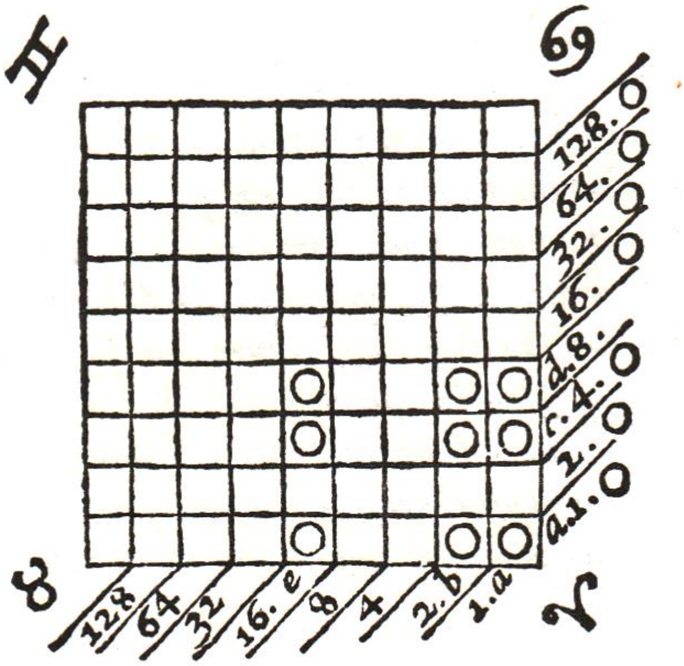

Figure 3

Napier concludes his chapter on multiplication with the following good advice:
It is much easier to follow all this using movable counters on a large board than from the ones immovably printed on the board in the diagram. You should therefore use the former when learning how to do it.

An ordinary chess or draughts board, with buttons or coins as counters, suffices for this example. The margins can be given labels 1…128 for convenience in whatever way is practical. Note that Napier's more elaborate example of multiplication of 1206 x 604 which requires a 20 x 20 chequer-board is treated in detail in Havil 2014. Particular attention is given there to dealing with the necessary abbreviation as multiple counters belonging to the product accumulate on the margin.
After gaining a good understanding of how multiplication works on the board it is relatively easy to understand procedures for division and taking the square root. The basic idea, in both cases, is to express the dividend (in the case of division), or the number of which we want the square root, on the righthand margin, then to set up on the board a suitable configuration which will exhaust – as far as possible – the value on that margin. For division we set up the dividend on the right margin and then successively decompose the dividend into segments that are of the same value as a multiple of the divisor. We shall follow Napier's first example which is to divide 250 by 13. Lay out counters on the right margin at *bdefgh* for the dividend (250) and *acd* on the lefthand margin for the divisor (13), see Figure 4.

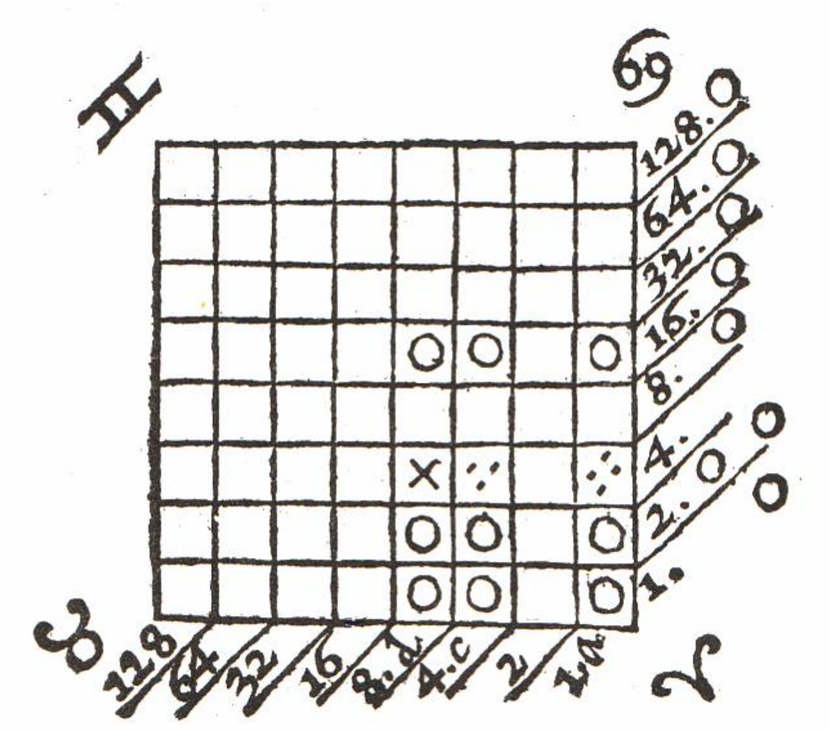

Figure 4

Moving (in imagination) by the rook's move from the 8 of the divisor and at the same time moving by the bishop's move from the 128 of the dividend the lines of movement intersect at a place where we assemble a segment with the same shape as the divisor. As the reader can see from Figure 4 it is a segment corresponding to 16 times the divisor. In imagination move the segment off the board subtracting their values from the dividend. Now we should remove the counters for 128, 64 and 16 from the dividend to see clearly that there remains 32, 8 and 2. Try and repeat the process following the diagonal from 32 on the right margin to the direct movement from the 8 on the left margin and place a segment like the divisor. It is marked in Napier's diagram (Figure 4) with a '+' and some patterns of dots. It cannot be used because its value (52) is greater than what remains of the dividend (42). We therefore drop down a square closer to the divisor and again place a similar segment which points to the 2 of the dividend. This is what Napier calls the first *congruous segment* (one small enough to subtract from what remains of the dividend). The one counter at 32 is worth two at 16, one of which is now removed by the head of the new segment. The other counters of the segment remove the 4 and the 2. There is still room for another segment pointing to 1 and by similar manoevres in subtracting this we are left finally with counters on 2 and 1in the right margin with segments pointing at 16, 2, 1. The quotient is 19 with remainder 3.
We are pleased to point out that Havil 2014 deals in considerably greater detail with three examples of division, with nicely redrawn diagrams, illustrating the snags and subtleties that may arise in the division procedures. He does not, however, deal with the extraction of a square root to which we turn now.

Extraction of a Square Root

Place counters in the right margin to represent the number of which we want the square root. Starting from the vertex ♈ of the board, move upwards towards the diagonally opposite vertex ♊. Each successive square is 4 times the value of its predecessor and all of them are (by symmetry) perfect squares. See Figure 5, the perfect squares are marked on the board by dots. Place a counter on the highest square of this diagonal that can be subtracted from the number on the righthand margin. This is the so-called 'head of the gnomons'. A 'gnomon' is this context means an L-shape of counters, with equal arms, which fits around a square, or a square of squares, to make a larger square. And that larger square must consist of identical segments. The first gnomon has 3 squares, the next ones have 5, 7, 9, etc. The counters need not be adjacent to each other on a gnomon but the square produced must be the product of identical numbers from each margin.

We follow Napier's procedure in finding the square root of 1,238. On the right margin we place counters at 1024, 128, 64, 16, 4 and 2.

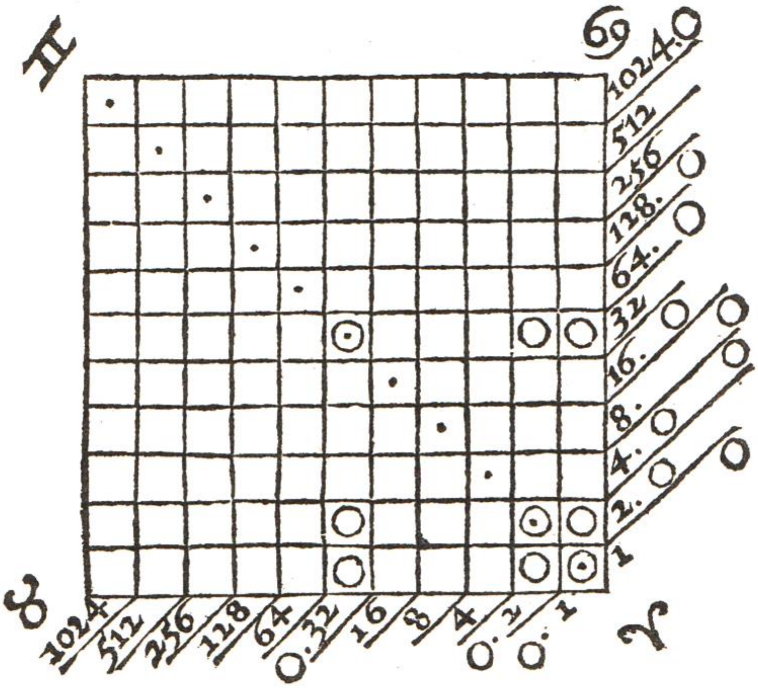

Figure 5

The head of the gnomons will be on the main diagonal at value 1024, we place and leave a counter there but subtract its value from the right margin. The next possible gnomon to place on the board will clearly be too high in value to subtract from the margin. We pull it away but in such a way that its (three) counters form a square with the counter of value 1024. Even the next two times we do this it has too high a value. Finally, at the level in which two counters have value 64 and one has value 4 it is what Napier, of course, called a 'congruous gnomon'. On removing the value of this gnomon there are still counters on the right margin at 64, 16 and 2. And there is still room for another gnomon with five counters to fit and be subtracted. This leaves counters on 8, 4, 1 . The segments of the square of gnomons point to 32, 2 and 1 so the square root of 1,238 is 35 with a remainder of 13.

Reflections

The broader picture

It has often been noted that Napier's reputation in his lifetime was based firmly on his antipapist tract (Napier, 1593) while a century later it was equally firmly based instead on his discovery and publication of logarithms. In a masterly two-page summary of Napier's life and work (Fauvel, 2000) an intriguing proposal was made to connect these two very different 'reputation-makers'. And the connection was explicitly extended to the chessboard abacus. Once Napier had adopted interpretations for items in the Book of Revelation such as trumpets, the seals, the candlesticks etc, the ongoing interpretation was, Fauvel suggested, a kind of mapping between two continua – the historic time-line from the birth of Christ onwards, and the narrative time-line in St John's vision of the Apocalypse. The mapping could allow us to use information from the past to make prophecy about the future. The key idea Napier brought to define logarithms was, similarly, a mapping between two continuous movements: one where the distances moved in equal times were in arithmetic progression, in the other they diminished in geometric progression. Information from one could be used to make deductions in the other. Wisely Fauvel cautions that*, It might be ill-advised to push this parallel too far,* but he does extend the analogy to the chessboard abacus where he remarks:
[…] the process of converting into different numbers (decimal numbers converted into binary)*, carrying out your operation, and then coming back, is structurally the same as the logarithmic procedure; and indeed, one might argue, of his theological procedures.*

Originality and relation to binary positional arithmetic

It should be clear from what we have described above that the similarity of what Napier is doing with his location arithmetic on a chessboard to what had been customary for centuries on the counting-board or the chequered-board was quite superficial. His concept of location numeral is unprecedented in the subtlety of referring simultaneously to a string of 'immovable' symbols and to a configuration of movable counters. His location arithmetic beautifully blends the direct and diagonal movements characteristic of the rook and bishop (respectively) in chess to represent multiplication and division of counter aggregates on the margins - through simple movements on the board – back to results on the margins. Although there is no explicit binary positional notation here it is always just below the surface. The numeral *abe* could be written *ab..e* with dots for omitted letters. Then writing this in reverse (highest power of 2 first) and putting a 1 where there is a letter and a 0 for a dot we obtain 10011 the standard positional binary numeral for 19. Similarly *acd* for 13 yields binary1101 and the product in usual 'long multiplication' style is

```
   10011
  x 1101
   10011
 1001100
10011000
11110111
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The repeated segments 10011 correspond then exactly to the repeated segments of counters in Fig 3.

It is ironical that Thomas Harriot (1560 – 1621) was doing exactly such positional binary arithmetic in his unpublished notes described in Shirley1951. It is hard to imagine though that Napier was not well aware of the possibility of such positional arithmetic. It's possible he deemed it not so useful to his readers: instead perhaps he saw himself as putting what we might now call a 'user interface' – the moveable counters - on top of the binary representation. This made it more fun, and easier to use - no need to write symbols at all. So to suggest the binary positional notation is 'just below the surface' may be quite misleading – he may have deliberately *added* the 'surface' of counter movements to render the arithmetic easier to carry out.

Relation to logarithms

The work on logarithms is treated in other parts of this volume. There is some internal evidence that the *Rabdologia* was being composed well before either of the works concerning logarithms were published. However that may be, it is clear that in spite of the 'game-like' spirit of his preface to the *Location Arithmetic*, Napier's vision for calculation, even here, was serious and large-scale. Two evidences for this are the board diagram in Figure 6 which is a square of side 24 which appears in his Chapter 6, and the reference in his Chapter 1 that if,
[…] you want to work with large numbers (such as sines, tangents and secants) make the rod […] accommodate 48 counters […] the last being 140,737,488,355,328. Such a rod will calculate all numbers less than double this (i.e. less than 281,474,976,710,656).

Note that the righthand margin of the 24-square in Figure 6 - albeit bent at the vertex ♋ - is labelled clearly up to the 47th cell (and is indeed half of what Napier claims is the 48th number!).
We remark here on three ways in which there are structural similarities between Napier's work on logarithms and the work on the chessboard abacus. Firstly, and most obvious, is the adoption of a new kind of number (or numeral) in the place of ordinary numerals. So ordinary numbers are replaced by artificial ones (logarithms) and then after manipulation restored to the ordinary kind. In location arithmetic the ordinary numeral has to be replaced by an equivalent in location numerals, then manipulated via counters on the board, and finally restored to ordinary numerals. Secondly, there is – to modern eyes – a curious lack of prominence of the *base* of the logarithms (awaiting Briggs explicit adoption of base 10), equally there no mention of *base* 2 in writing the strings encoding the powers of 2. This is merely a hindsight matter, with wider familiarity and frequency of examples the concepts have emerged as significant when initially they were only dimly, if at all, perceived. Finally both the definition of logarithm, and the definition of the procedures for multiplication and division, depend on certain movements - albeit imagined movements. The motion of a particle P that is slowing down so its velocity is proportional to the distance yet to travel (say PQ) is compared with the motion of a particle L travelling at constant velocity. In equal time intervals the distances the former particle travels are in geometric progression, the distances of the latter particle are in arithmetic progression.

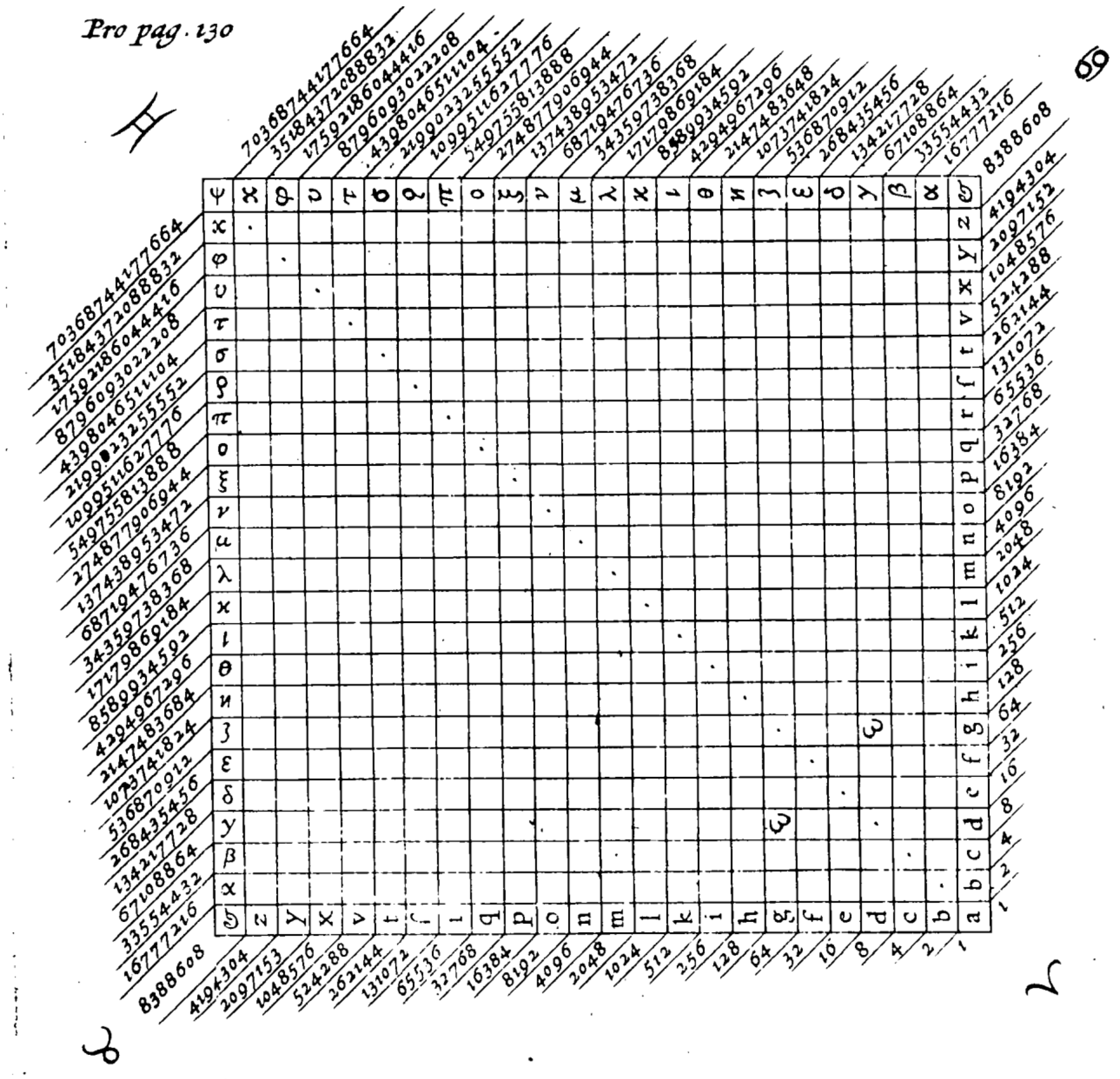

Figure 6

Napier defines the logarithm of PQ to be the distance L has travelled from the time that P started. So the 'movements' here do not actually need to take place – they are only to set up a relationship. Similarly with the movements involved in multiplication and division in location arithmetic – they are only aids to the imagination in finding intermediate values. In each case it is a kind of 'pseudo-movement' and does not justify any connection with analogue computing.

Relation to the abacus

In location arithmetic, in addition to the pseudo-movement to which we have just referred, there is a real and vital movement of the counters which constitutes the progress of the calculation and records the state of the calculation at any stage. It is this movement, closely comparable to the movement of beads on wires of the traditional abacus, that of course justifies our term 'chessboard abacus'. The Latin 'abacus' (or cognate) appears frequently in the original text (for example, in the titles of Chapters 6,7,8), but it has been rendered, no doubt most suitably, in such contexts as 'board' in Richardson's translation. The human contribution to the computation is more sophisticated here than on the standard abacus – especially in the 'trial and error' involved in subtracting segments or gnomons in division or square-rooting. But the role of the counters is equally vital in mediating the human contribution.

As with the abacus, the state of the computation in location arithmetic – in particular the various intermediate states - is always interpretable by the human user. This is often not the case in modern procedural programming where auxiliary variables are introduced having only internal interpretations.
Again in common with the abacus the configuration of counters, at various stages, acts as a kind of 'system agency' that cues the human agency in a very close two-way relationship. For example the completion of the rectangle on the board for the multplication of two numbers has to recognised by the human and acts as a cue to sweep the counters off the board and into a margin for further processing jointly by the sequence of cells provided and the human agency.

Relation to modern computing

A major thread in the development of electronic computing since the 1940's is achieving the goal of 'efficient automation'. It is epitomised in the title of Turing's first stored-program computer design - ACE (Automatic Computing Engine) – see Turing 1946. Occasionally there have been complementary visions such as that of one of the leading computer scientists of the 1960's, JCR Licklider:
The hope is that, in not too many years human brains and computing machines will be coupled together very tightly and that the resulting partnership will think as no human brain has ever thought (Licklider 1960)

After well over 50 years this has not happened and still does not seem likely. For many applications it is also clearly not desirable. But for some areas, such as decision support, design thinking, stimulating and supporting imagination, game design, crisis management, etc it seems indispensible. Unfortunately the dominance of 'automation is everything' has shaped so-called computational thinking and shaped available technology itself to make human intervention very difficult unless it has been preconceived.
Napier's chessboard abacus is a reminder of an age when close and continuous collaboration between human and technical artefacts was a normal expectation. The Empirical Modelling project at Warwick promotes a view of computing that supports such collaboration 'on-the-fly' as opportunity or need arises; we have called it 'human computing'. This is complementary to the traditional 'Turing machine' view emphasising abstraction and algorithms – which we also admire and endorse. The idea is that in adopting our broader fundamental concepts of observables, dependency and agency for our work in computing we are closer to the way humans think, and make sense of things. Our environments, and our software for building models, make use of the same concepts and principles which are strongly derived from experience.

A consequence of the 'experiential' approach is that our models have an unusual openness and flexibility. As an illustration for this article there is a link on the webpage (URL see Ref. Empirical Modelling) to a page, Napier's Chessboard Abacus, where there is further information and an illustrative interactive model of Napier's devices of rods and boards for arithmetic.

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